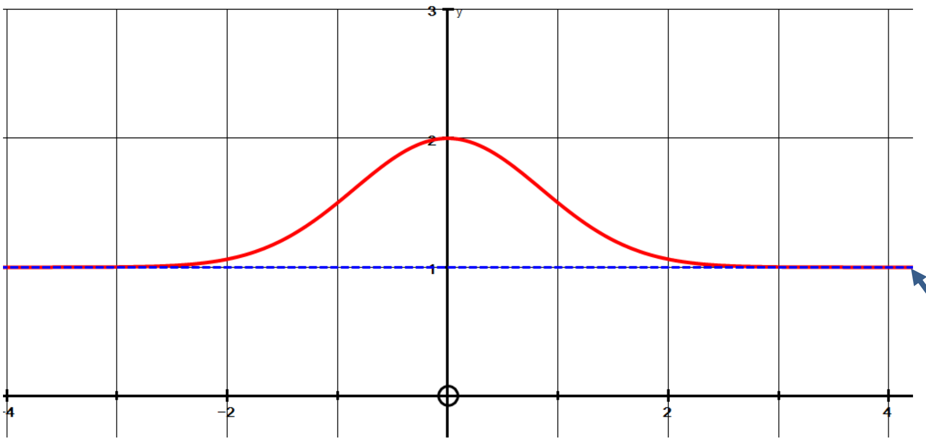
A Brief Summary of ASYMPTOTES.

Def: Asymptote: a line that draws increasingly nearer to a curve without ever meeting it.

There are a basically three types of asymptotes: *horizontal*, *vertical and oblique.*

1. The graph $y = 1 + 2^{-x^2}$ or $y = 1 + \frac{1}{2^{x^2}}$ has **ONE *horizontal*** *asymptote.*



When ***x*** is "large" (meaning in this case, ***x*** > 3 and ***x*** < -3)

the value of 2^{-x^2} or $\frac{1}{2^{x^2}}$ gets very close to 0

eg if ***x*** = 4 then 2^{-4^2} or $\frac{1}{2^{4^2}}$ ≈ 0.000015

so we say that ***y*** → 1 + 0 = 1

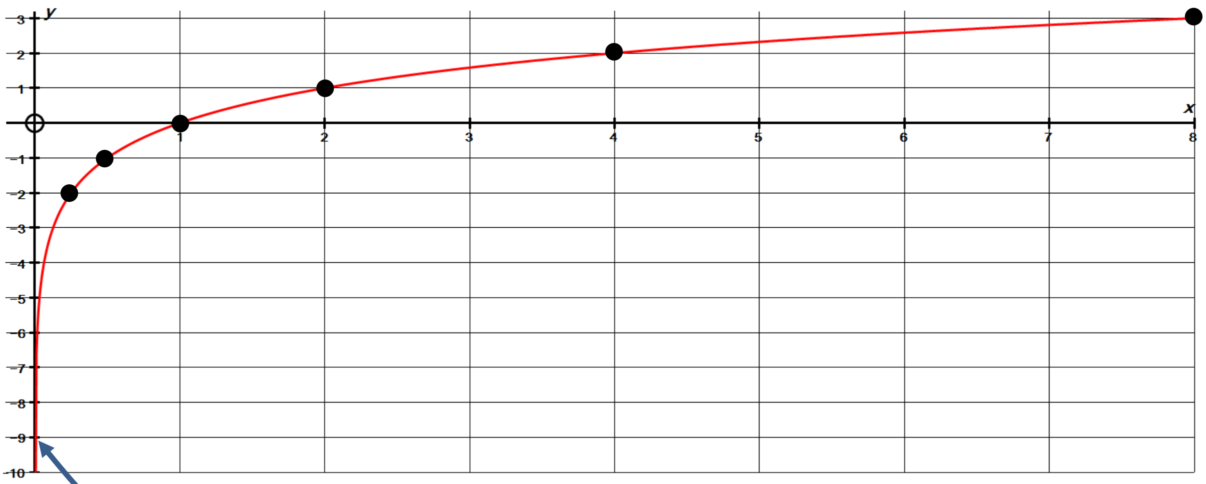
and we say the ***horizontal asymptote*** is the line ***y = 1***

2. A graph can also have just **ONE *vertical asymptote.***

Consider ***y = log₂x***

Typical points are:

x	1	2	4	8	½ = 2^{-1}	¼ = 2^{-2}	⅛ = 2^{-3}	2^{-4}	2^{-10}
y = log₂x	0	1	2	3	– 1	–2	–3	–4	–10



Clearly, the vertical asymptote is the ***y*** axis (*or the line **x = 0***)

3. Consider the graph of $y = \dfrac{4}{(x-1)}$

If x is large (in this case let x be over, say 40) eg I will choose $x = 41$
then $y = \dfrac{4}{40} = \dfrac{1}{10}$

and going in the negative x direction I will choose $x = -39$
then $y = \dfrac{4}{-40} = -\dfrac{1}{10}$

We can say that for large positive x values and "large" negative x values $y \to 0$
and we say the ***horizontal asymptote*** is the line $y = 0$

However, when x is close to 1 on either side, the y value gets big!

Approaching x = 1 from the right:

If x = 1.1 then y = 40
If x = 1.001 then y = 4000
If x = 1.000001 then y = 4 000 000

Approaching 1 from the left:

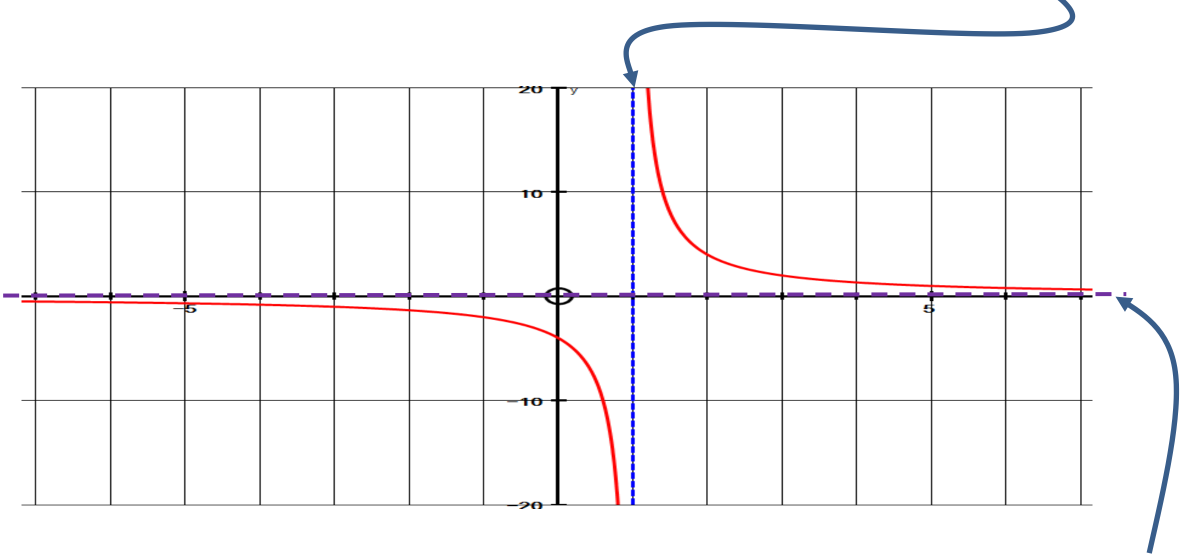
If x = 0.9 then y = – 40
If x =0.999 then y = – 4000
If x = 0.999999 then y = – 4 000 000

Clearly there is <u>no limit</u> to what the *y* value can reach!

There actually is **no definite number** that y equals when $x = 1$
We say that y is INFINITE (which could be referred to as non finite).

As x →1 from the right we say y → + ∞
As x →1 from the left we say y → – ∞

The graph approaches a ***vertical asymptote*** and its equation is $x = 1$



horizontal asymptote y = 0

4. It is quite common to see graphs with **vertical** and **horizontal** asymptotes:

eg $y = \dfrac{2x - 4}{x - 1}$

The vertical asymptote is when the denominator = 0 ie if x = 1

To find the horizontal asymptote we first do a long division:

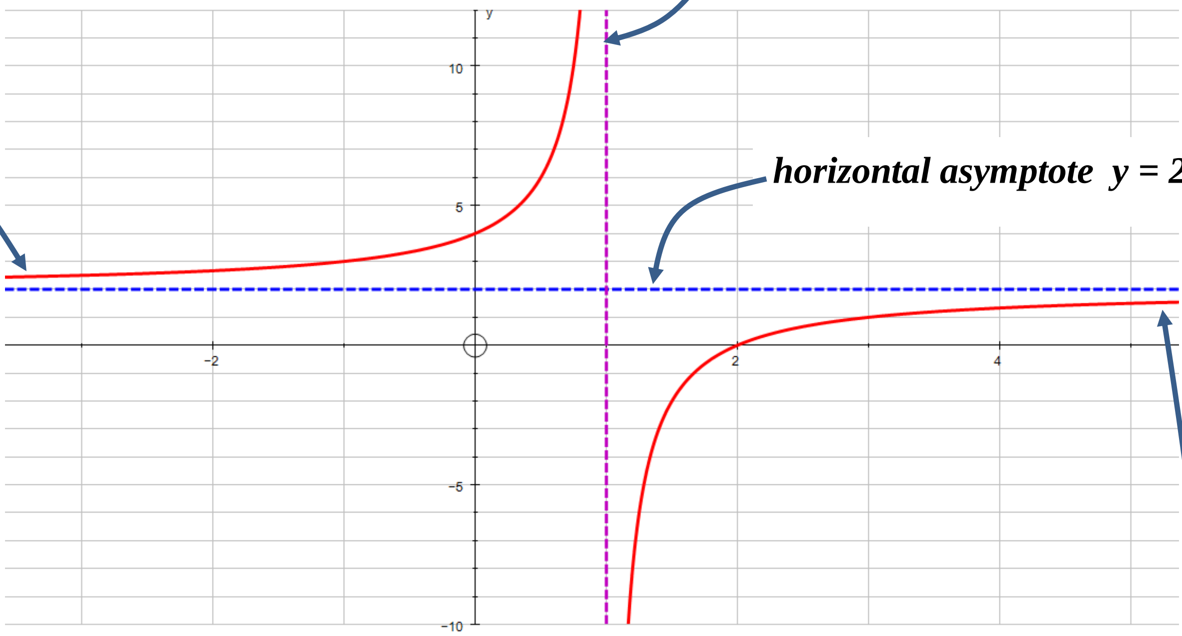
$$\begin{array}{r} 2 \\ x-1 \overline{) 2x-4} \\ \underline{2x-2} \\ -2 \end{array}$$

so $y = 2 - \dfrac{2}{x - 1}$

When x is large, the term: $\dfrac{2}{x-1}$ ***approaches 0***

so the horizontal asymptote is y = 2

vertical asymptote x = 1



NB A useful idea to remember is if $y = 2 - \dfrac{2}{x-1} = 2 -$ ***"a bit"***

then for large and positive x values the y value is "a bit less" than 2
so the curve is UNDER the horizontal asymptote $y = 2$

(But for "large" x values which are negative $y = 2 +$ ***"a bit"*** so the left hand side of the graph is ON TOP of the asymptote $y = 2$)

5. The following graph has a **vertical** asymptote and an **<u>oblique</u>** one:

$$y = \frac{x^2}{(x-1)}$$

Firstly, the vertical asymptote is when the denominator is zero so the equation is x = 1

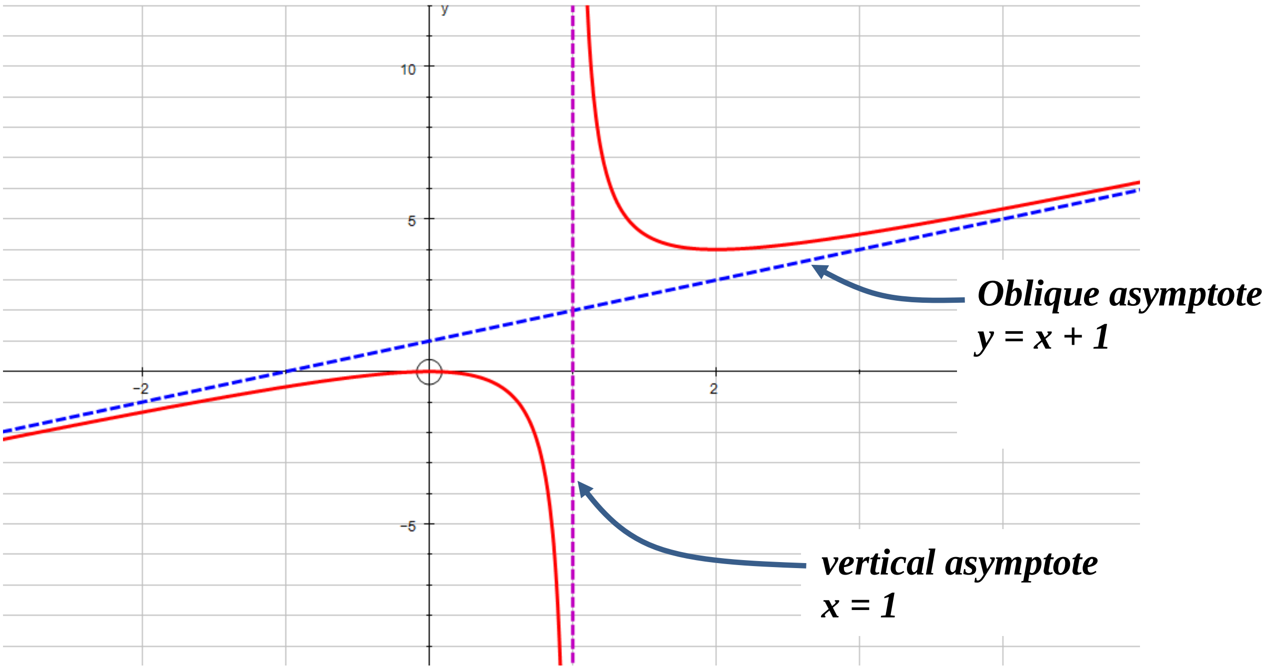
To find the OBLIQUE asymptote we first do a long division:

$$\begin{array}{r|l} & x + 1 \\ \hline x - 1 & x^2 + 0x + 0 \\ & \underline{x^2 - x} \\ & \quad x \\ & \quad \underline{x - 1} \\ & \qquad 1 \end{array}$$

so $y = x + 1 + \dfrac{1}{x-1}$

Since $\dfrac{1}{x-1} \to 0$ ***when x is large***

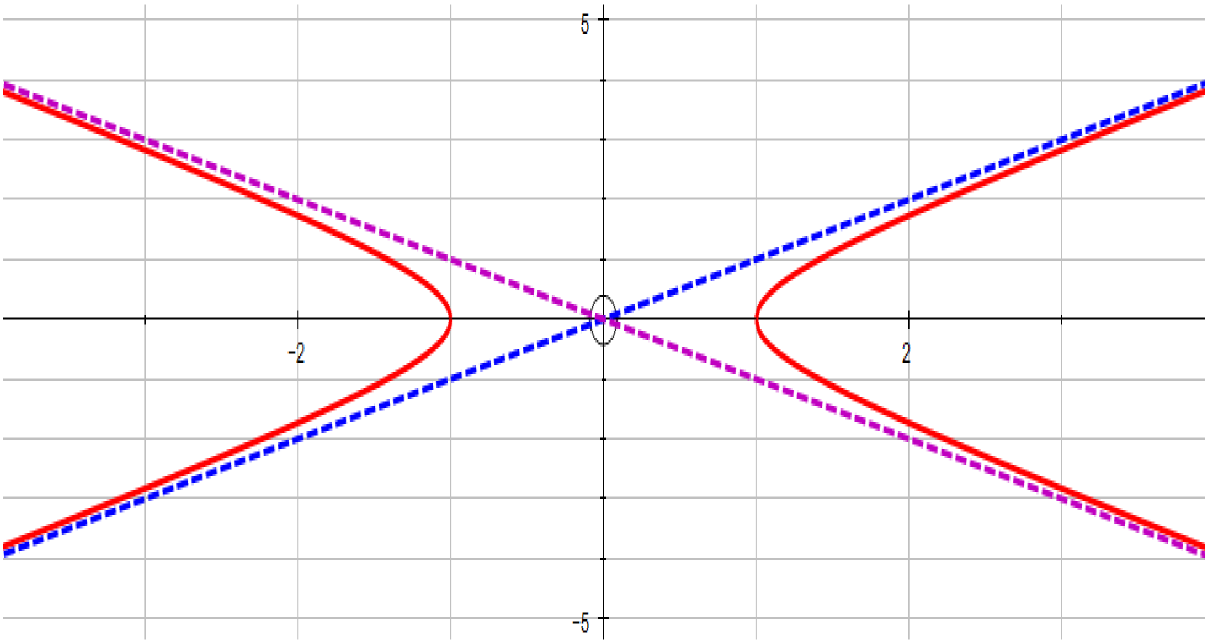
the curve is approximately y = x + 1
This means the curve approaches this line which is referred to as <u>oblique</u> because it is neither vertical nor horizontal.



6. A curve can have **TWO** *oblique asymptotes:*

Consider the curve: $y^2 = x^2 - 1$

If x is large then the "1" becomes negligible so the equation reduces to:
$y^2 \approx x^2$ ***so that the asymptotes are*** $y = \pm x$ ***as shown below:***



7. I had never come across a curve with a **horizontal** asymptote and an **oblique** one. **So I made up one!**

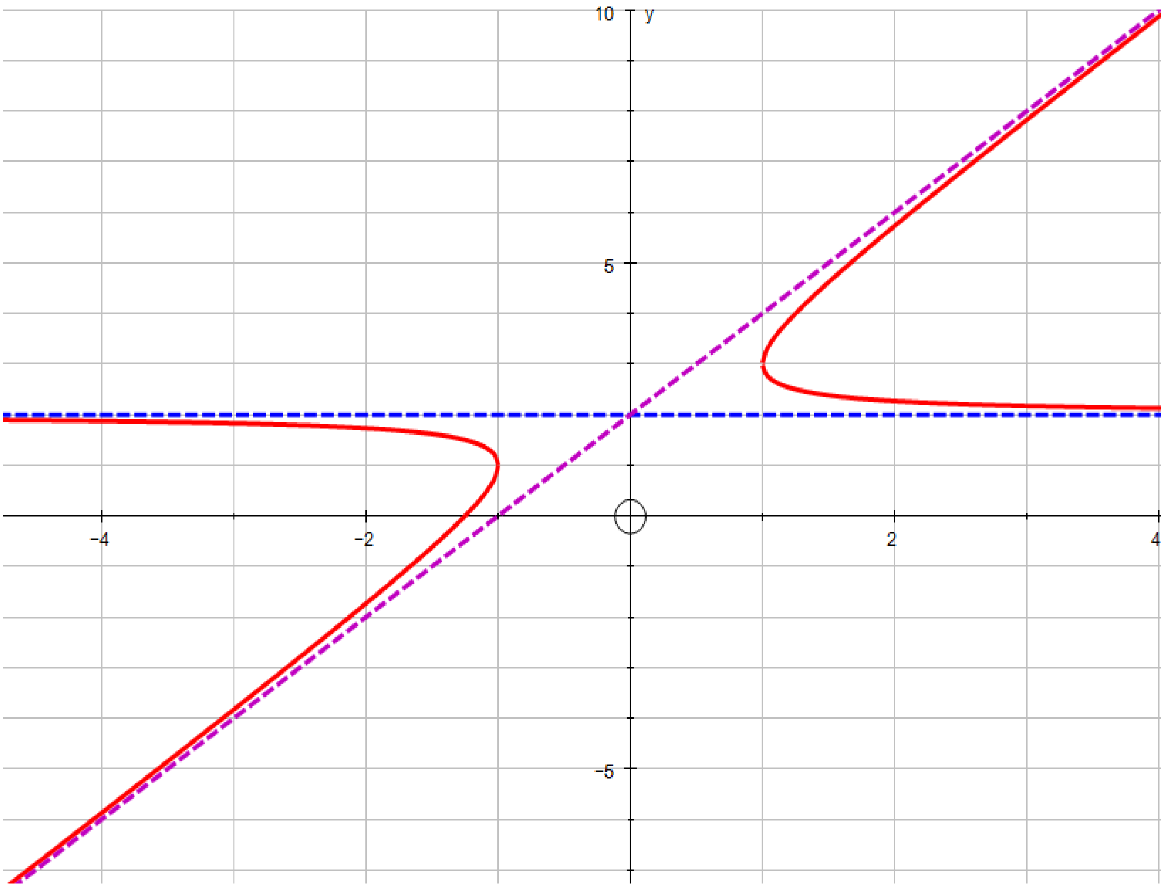
Consider $(y - x - 2)^2 = x^2 - 1$

As in number 6, if ***x and y are*** large then the "***1***" becomes negligible so the equation reduces to: $(y - x - 2)^2 \approx x^2$

So $y - x - 2 = +x$ ***or*** $y - x - 2 = -x$

$y = 2x + 2$ ***or*** $y = 2$

The equation: $y = 2x + 2$ ***is the <u>oblique</u> asymptote and***
The equation: $y = 2$ ***is the <u>horizontal</u> asymptote.***

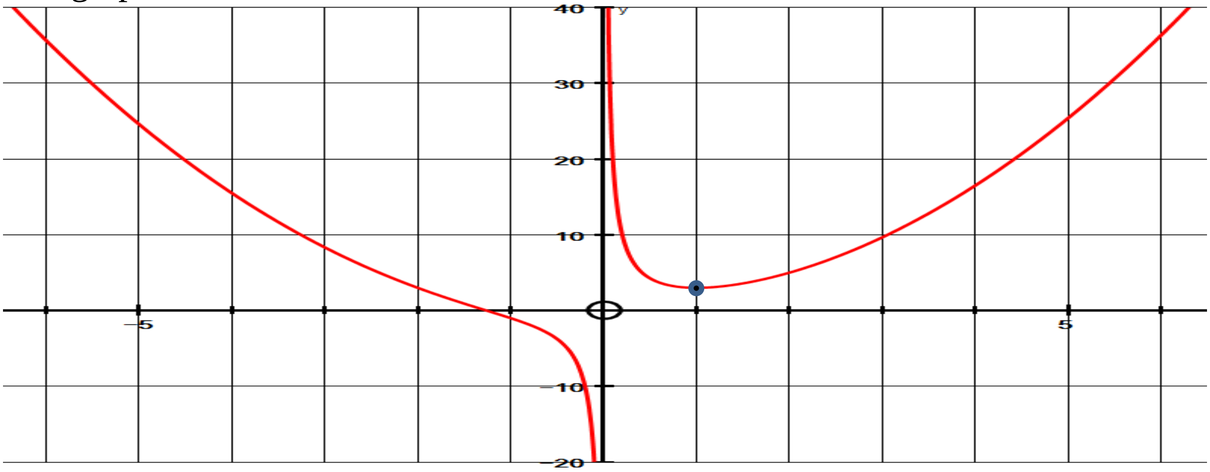


8. Although an asymptote is really a LINE which a curve approaches, we can also have curves approaching **curves**!

For instance, consider $y = \dfrac{x^3 + 2}{x} = x^2 + \dfrac{2}{x}$

The denominator is zero if $x = 0$ so this is the ***vertical asymptote.***

The graph looks like this:



Just as a matter of interest $y' = 2x - \dfrac{2}{x^2} = 0$ ***at the minimum point.***

$$2x = \frac{2}{x^2}$$

$$\text{so } x^3 = 1$$

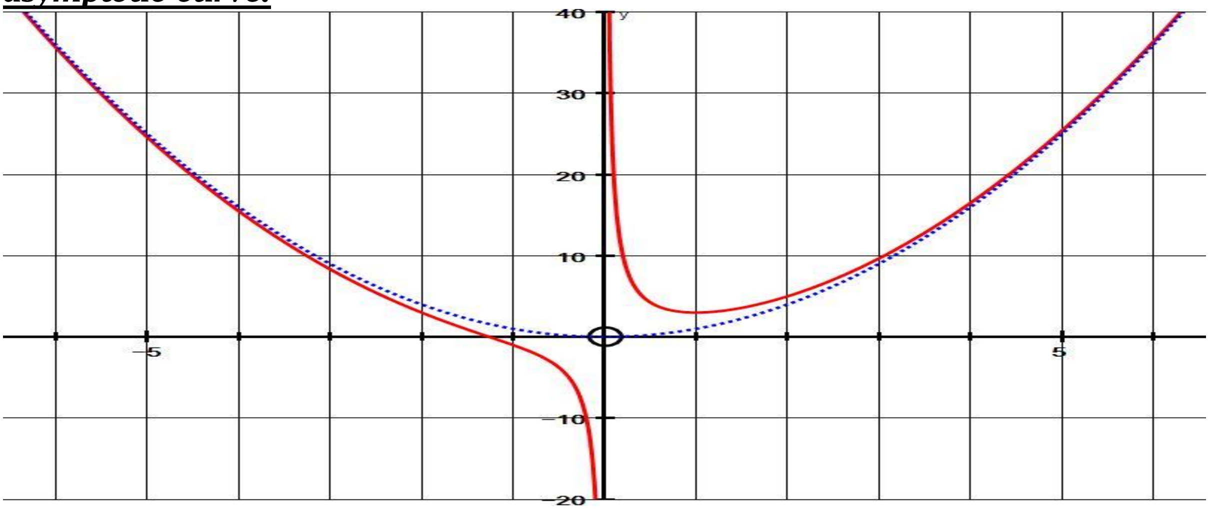
$$x = 1 \text{ and } y = 3 \quad \textbf{min point is (1, 3)}$$

However, if we examine: $y = x^2 + \dfrac{2}{x}$

we see that, if x is large, $\dfrac{2}{x}$ ***becomes negligible*** (*if* $x = 200$ *then* $y = 200 + \dfrac{1}{100}$)

so that the equation is approximately $y = x^2$

If I superimpose this parabola on the above graph we can see that it acts as an ***asymptotic curve.***



9. Some people misunderstand the dictionary definition of an asymptote.
(A line that draws increasingly nearer to a curve ***without ever meeting it****.)*
Sometimes a curve can pass <u>through</u> an asymptote!
Remember, an asymptote is what a curve approaches for "large" values of x.

Consider $y = \dfrac{(x-2)(x-4)}{(x-1)(x-3)} = \dfrac{x^2 - 6x + 8}{x^2 - 4x + 3}$

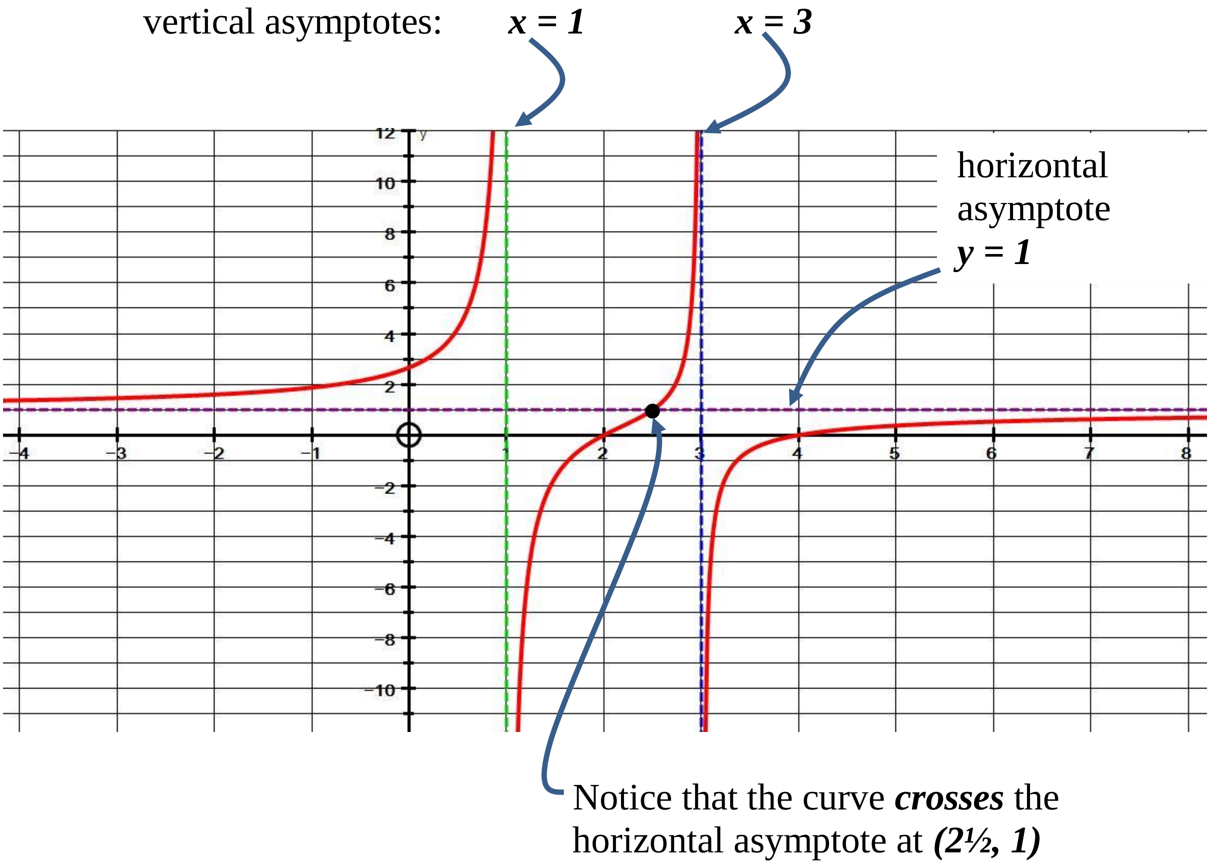
The denominator is zero when $x = 1$ and 3 so the vertical asymptotes are $x = 1$ ***and*** $x = 3$

By long division:

$$\begin{array}{r|l} & \quad 1 \\ x^2 - 4x + 3 & x^2 - 6x + 8 \\ & \underline{x^2 - 4x + 3} \\ & \quad -2x + 5 \end{array}$$

So $y = 1 \quad + \quad \dfrac{-2x + 5}{x^2 - 4x + 3}$

and when x is large $y \approx 1$ since $\quad \dfrac{-2x + 5}{x^2 - 4x + 3} \quad \to 0$



Note: If $y = \dfrac{x^2 - 6x + 8}{x^2 - 4x + 3} = 1$ *then* $x^2 - 6x + 8 = x^2 - 4x + 3$

so $5 = 2x$

and $x = 2\frac{1}{2}$, $y = 1$

10. We need to be very careful with equations like number 9 because there may be a turning point involved as with the following:

$$y = \frac{x(x-4)}{(x-1)(x-3)} = \frac{x^2 - 4x}{x^2 - 4x + 3} = 1 - \frac{3}{x^2 - 4x + 3}$$

This has the same vertical asymptotes $x = 1$ ***and*** $x = 3$ and the same horizontal asymptote $y = 1$ but the graph is quite different!

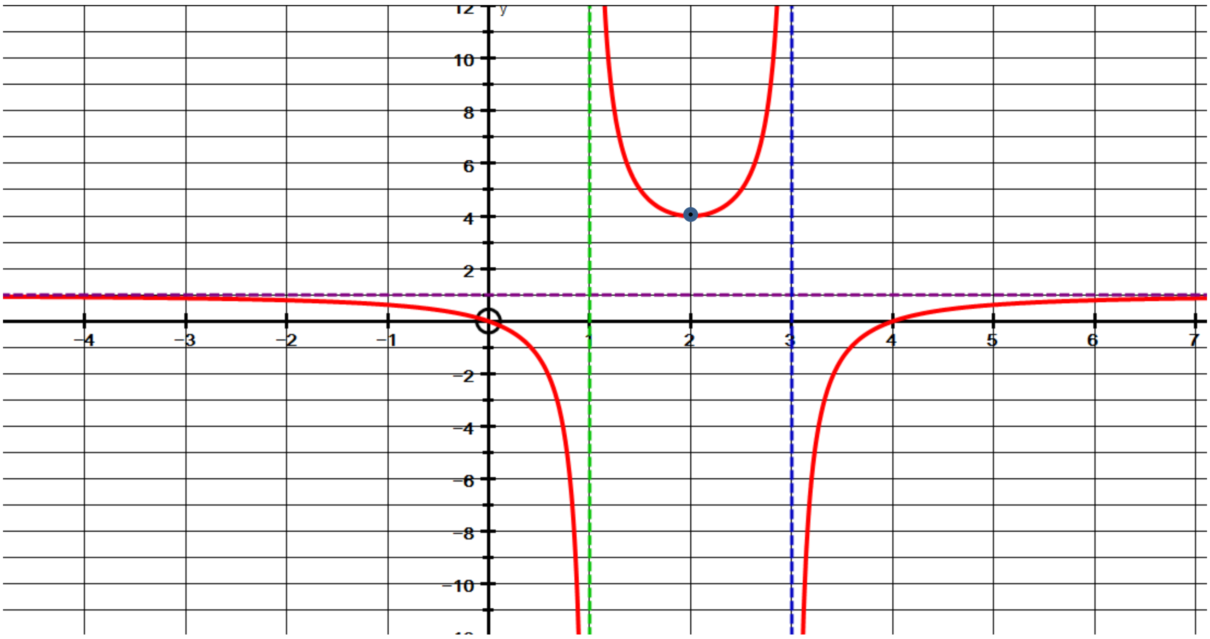
$$y' = \frac{(x^2 - 4x + 3)(2x - 4) - (x^2 - 4x)(2x - 4)}{(x^2 - 4x + 3)^2} = 0 \text{ at max/min points}$$

$$2x^3 - 4x^2 - 8x^2 + 16x + 6x - 12 - 2x^3 + 12x^2 - 16x = 0$$

$$6x - 12 = 0$$

$$x = 2 \text{ and } y = 4$$

minimum point is at (2, 4)



NB A useful idea to remember is if $y = 1 - \frac{3}{x^2 - 4x + 3} = 1 -$ ***"a bit"***

then for large and positive x values the y value is "a bit less" than 1
so the curve is UNDER the horizontal asymptote $y = 1$
(In this case it is also true for "large" x values which are negative.)

11. There can be a "trap" in the method if we are not fully aware of it.

Consider $y = \frac{x^2 - x - 6}{x - 3}$

We may be ***expecting a vertical asymptote with equation x = 3*** because this makes the denominator zero.

It is true that a denominator of zero makes the number infinite but in this case it makes the numerator zero as well, producing $\frac{0}{0}$ which is not infinite but it is called ***indeterminate***. (ie cannot be determined or found)

If we examine $\frac{x^2 - x - 6}{x - 3}$ and factorise the numerator (ie top line)

we get $\frac{(x-3)(x+2)}{(x-3)}$

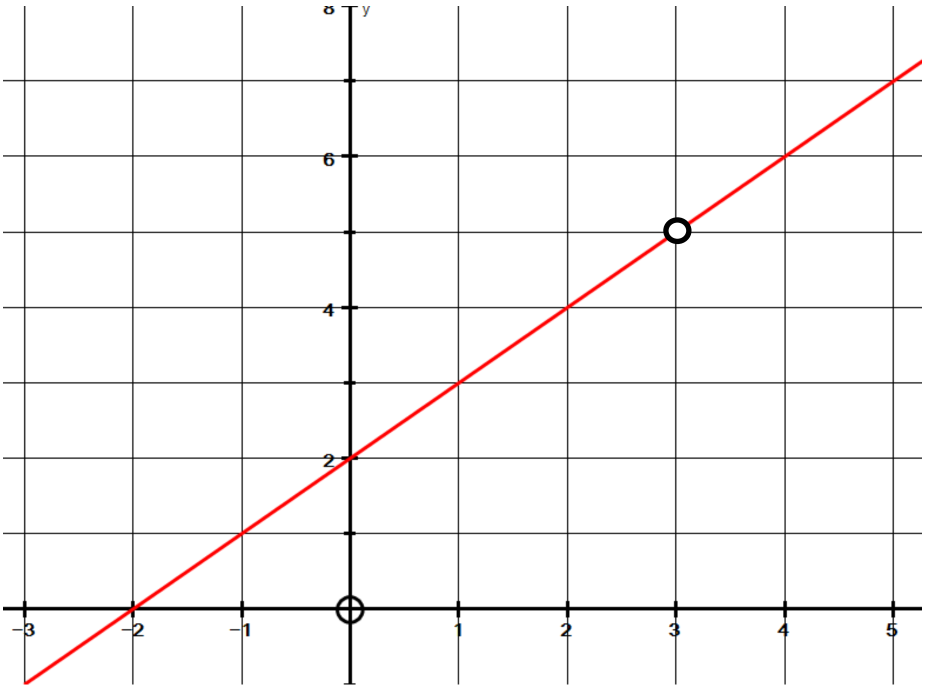
Now this ***does*** equal $(x + 2)$ if we cancel the $(x - 3)$ terms but we can only do this if $x \neq 3$ because this would involve $\frac{0}{0}$

The conclusion is $y = \frac{x^2 - x - 6}{(x-3)} = \frac{(x-3)(x+2)}{(x-3)} = x + 2$ (if $x \neq 3$)

This means that the graph $y = \frac{x^2 - x - 6}{x - 3}$

is just a line graph with a HOLE in it at x = 3.

We show the "hole" by putting an "open circle" at the point (3, 5)



12. A very unusual graph is $\boldsymbol{y} = \dfrac{4x}{x^2 + 4}$

The denominator $x^2 + 4$ cannot be zero so there are ***no vertical asymptotes***.
However if x is quite large (in the positive and negative directions)
the expression $\dfrac{4x}{x^2 + 4}$ ***does approach zero.***

eg if $x = 100$ then $y = \dfrac{400}{160004} \approx 0.0025$

Differentiating: $y = \dfrac{4x}{x^2 + 4}$

we get $y' = \dfrac{(x^2 + 4)4 - 4x.2x}{(x^2 + 4)^2} = 0$ ***at max/min points***

Solving: $4x^2 + 16 - 8x^2 = 0$

$4x^2 = 16$

$x^2 = 4$

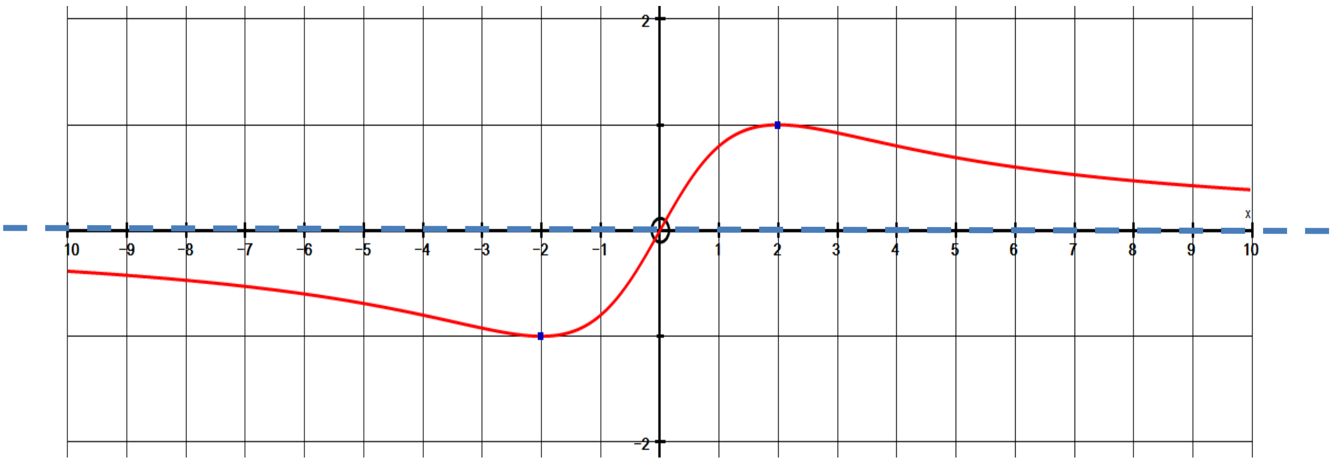
$x = \pm 2$

Testing gradients on either side of ±2

x	1	2	3
y'	+	0	–

x	–3	–2	–1
y'	–	0	+

This is the graph with a max at (2, 1) and a min at (– 2, – 1) and approaching y = 0 in both directions. (note: it <u>crosses the asymptote</u> at x = 0)



Special Notes for "HYPERBOLAPHOBES"!

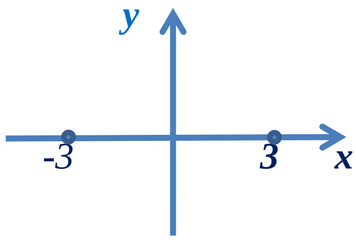
13. Consider the Hyperbola: $\dfrac{(x-6)^2}{9} - \dfrac{(y+1)^2}{4} = 1$

I usually tell students to first draw $\dfrac{x^2}{9} - \dfrac{y^2}{4} = 1$

METHOD: Put $y = 0$ so $\dfrac{x^2}{9} = 1$

so $x^2 = 9$

and $x = \pm 3$

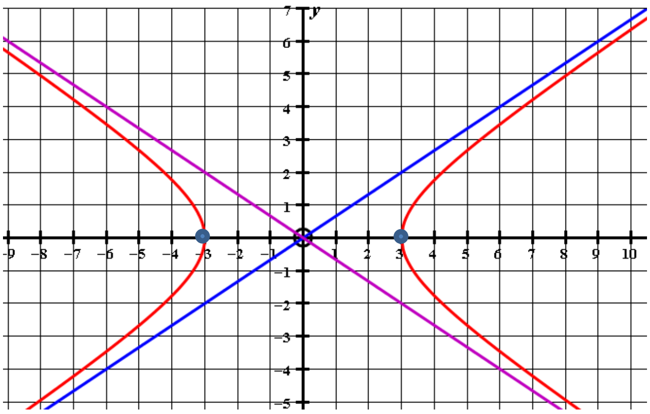


Asymptotes : write $\dfrac{x^2}{9} - \dfrac{y^2}{4} = 1$

as $\dfrac{x^2}{9} = \dfrac{y^2}{4} + 1$ *or* $\dfrac{y^2}{4} + 1 = \dfrac{x^2}{9}$

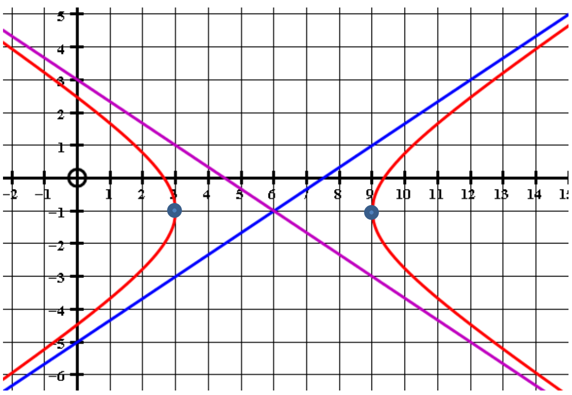
IF *x* and *y* are large then the "*1*" is negligible so:

$\dfrac{y^2}{4} \approx \dfrac{x^2}{9}$ *so that* $y^2 \approx \dfrac{4}{9}x^2$ *and the asymptotes are* $y = \pm \dfrac{2}{3}x$



Note: I never like to use the standard formulae for these asymptotes.
The above method emphasises logical thinking.

So to draw $\dfrac{(x-6)^2}{9} - \dfrac{(y+1)^2}{4} = 1$ *we just move it* ***along*** **6** *and* ***down*** **1**.



We need to move the asymptotes so that they now go through (6, – 1)

If $y = \dfrac{2}{3}x + c$ ***and*** $y = -\dfrac{2}{3}x + d$

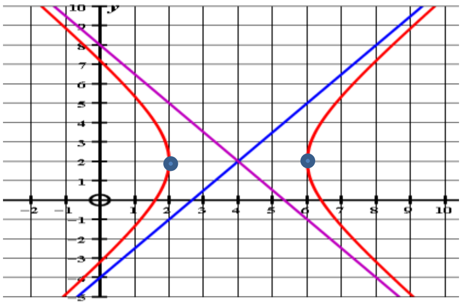
We subs $x = 6, y = -1$

so $c = -5$ ***and*** $d = +3$

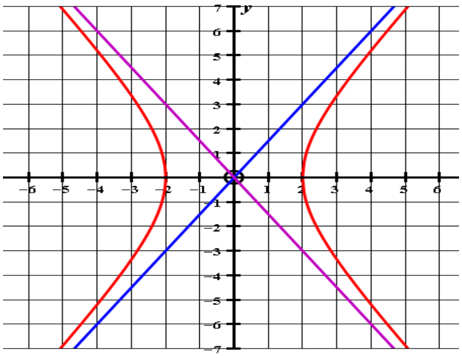
The equations of the asymptotes are:

$y = \dfrac{2}{3}x - 5$ ***and*** $y = -\dfrac{2}{3}x + 3$

14. AND we should be able to find the equation of an already drawn hyperbola:



Like before it is better to consider the hyperbola with its centre at the origin:



We need to be **sure** the type of equation in this case is :

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

and not $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ we need to find a and b.

we see the graph goes through ***$x = 2, y = 0$*** so substituting we get :

$\frac{4}{a^2} = 1$ so that $a^2 = 4$

The equation so far is ***$\frac{x^2}{4} - \frac{y^2}{b^2} = 1$***

For large x and y (the "1" is negligible) ***so*** ***$\frac{y^2}{b^2} \approx \frac{x^2}{4}$ so $y^2 \approx \frac{b^2 x^2}{4}$***

and so the asymptotes are ***$y = \pm \frac{b}{2} x$***

BUT WE CAN SEE FROM THE GRAPH THAT THE ASYMPTOTES ARE:

$$y = \pm \frac{3}{2} x$$

SO OBVIOUSLY ***$b = 3$*** and the equation is ***$\frac{x^2}{4} - \frac{y^2}{9} = 1$***

Finally, moving this along 4 and up 2 produces $\frac{(x-4)^2}{4} - \frac{(y-2)^2}{9} = 1$

And if we subs (4, 2) into $y = \frac{3}{2}x + c$ and $y = -\frac{3}{2}x + d$ respectively:

the asymptotes are $y = \frac{3}{2}x - 4$ and $y = -\frac{3}{2}x + 8$