**TRIGONOMETRIC FORMULAE.**

The formula sheet for New Zealand Calculus examinations contains a multitude of trigonometric formulae and despite teachers going to a lot of trouble carefully deriving these formulae, I guarantee that 99% of students do not remember where they have come from.

The most valuable and useful one concerns the trigonometric version of Pythagoras’ Theorem.

***sin2θ + cos2θ = 1***

The proof is so incredibly simple yet very few students actually can recall it when asked!

The first step is to let the hypotenuse be of length 1 unit:

Using the old SOH CAH TOA idea :

***sin θ = y = y***

***1 y 1***

***cos θ = x = x***

***θ 1***

***x***

Pythagoras’ Theorem is simply ***y2 + x2 = 12***

Substituting ***y = sin θ and x = cos θ*** we get (***sin θ)2 + (cos θ)2 = 1***

usually written as: ***sin2θ + cos2θ = 1***

Incidentally since ***tan θ = y*** and substituting ***y = sin θ and x = cos θ***

***x***

this is the best way to explain clearly WHY ***tan θ = sin θ***

***cos θ***

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I explained in a previous article that it is time to get rid of the reciprocal

trigonometric functions ( ***sec x, cosec x and cot x)***. They are a relic of past centuries and of no actual use whatsoever.

If we must actually give our calculus students the derivatives of ***sin x, cos x*** and ***tan x*** on the formula sheet then we should just give them the following version:

|  |  |
| --- | --- |
| ***y = f(x)*** | ***dy = f ꞌ(x)***  There should be no mention of ***sec x, cosec x*** and ***cot x*** whatsoever.  I think that the derivatives of functions such as ***y = (sin x) – 1***  should be simply worked out and left in terms of ***sin x*** and ***cos x***.  eg ***y = (sin x) – 1***  ***dy = – 1 (sin x) – 2 (cos x)***  ***dx***  = ***– cos x***  ***(sin x)2***  ***dx*** |
| ***ln(x)***  ***eax***  ***sin x***  ***cos x***  ***tan x*** | ***1***  ***x***  ***aeax***  ***cos x***  ***– sin x***  ***. 1 .***  ***cos2x*** |

A typical batch of very narrowly specialised formulae is:

**Products**

2sinA cosB = sin(A + B) + sin(A – B)

2cosA sinB = sin(A + B) – sin(A – B)

2cosA cosB = cos(A + B) + cos(A – B)

2sinA sinB = cos(A – B) – cos(A + B)

I suspect the **sole purpose** of these formulae is just so we can integrate things such as:

***= sin(8x) + sin(2x) + c***

***8* *2***

Frankly the time and effort needed to derive the above formulae, especially when many people do not even do a trigonometry assessment in their courses,

makes it not worthwhile to bother with them at all.

These integrals are hardly crucial to a first calculus course.

I think that the **Compound Angle** formulae are probably the only other useful ones which we actually need to use.



In order to understand how to multiply and divide **COMPLEX NUMBERS** in polar form, we need to recognise the formulae for sin(A + B) and cos(A + B)

Let *u = 1 cis(A) and v = 1 cis(B)*

**The PRODUCT *u×v***

*= (cos A + i sin A) × (cos B + i sin B)*

*= cos A cos B + i(sin A cos B + cos A sin B) +* ***i2*** *(sin A sin B)*

*=* ***cos A cos B – sin A sin B + i(sin A cos B + cos A sin B)***

*=* ***cos(A + B) + i sin(A + B)***

This means that *cis(300) × cis(450) = cis(30 + 45) = cis (750)*

Or *cis (π/6) × cis (π/4) = cis (π/6 + π/4 )= cis(7π/12)*

**The QUOTIENT *u***

***v***

*= (cos A + isinA) = (cos A + isinA)× (cos B* ***–*** *isin B)*

*(cos B + isin B) (cos B + isin B) (cos B* ***–*** *isin B)*

*= cos A cos B + sin A sin B* ***+*** *i( sin A cos B – cos A sin B)*

*cos2B + sin2B*

*=* ***cos(A – B) + i sin(A – B)***

This means that *cis (450) = cis(45 – 30) = cis ( 150)*

*cis (300)*

or *cis(π/4) = cis(π/4 – π/6) = cis(π/12)*

*cis(π/6)*

**Imagine how elated De Moivre would have felt discovering this!**

Also, in order to PROVE the derivative of ***y = sin x is yꞌ = cos x***,

instead of using the usual “right hand” form of the derivative :

*y ′ = lim f(x + h) – f(x)*

*h→0 h*

I like to use the “**two sided**” form of the derivative:

*x – h x x + h*

R

P

Q

*f(x – h)*

*f(x + h)*

*y = f(x)*

*2h*

The gradient of chord QR is a good approximation of the gradient of the tangent at P.

The gradient of the CHORD QR =*f(x + h) – f(x – h)*

*2h*

**The “two sided” version of the derivative is:**

Gradient at P *= lim* ***f(x + h) – f(x – h)***

*h→0* ***2h***

If ***y = sin x*** then *y ′ = lim sin(x + h) – sin(x – h)*

*h→0 2h*

This is much

easier for

students to

follow than

the traditional

method .

= *lim sin x.cos h* + *cos x.sin h* *– sin x.cos h +cos x.sin h*

*h→0 2h*

*= lim 2 cos(x) sin(h)*

*h→0 2 h*

*= cos x × lim sin(h)*

*h→0 h*

*=* ***cos x*** *×*  ***1*** ***if x* is in radians!**

( Or = *cos x* × **0.01745**  **if *x* is in degrees!** ) see below….

Consider : L = ***lim sin ( h )***

***h→0 h***

**USING DEGREES :**   **USING RADIANS :**

Let *h* = 0.0001 degrees Let *h* = 0.0001 rads

L ≈ sin (0.00010 ) L ≈ sin (0.0001 rad )

0.0001 0.0001

= 0.017453292 = .999999999… = **1**

So, if *y = sin (x degrees) So, if y = sin (x radians)*

***dy = cos(x) × 0.01745 dy = cos(x) × 1***

***dx dx***

Similarly for the derivative of *y = cos x*

Gradient at P *= lim* f(*x + h*) – f(*x – h*)

*h→ 0**2h*

If y = cos *x* then *y ′ = lim cos(x + h) – cos(x – h)*

*h→ 0**2h*

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= *lim cos x.cos h –* *sin x.sin h* *– cos x.cos h – sin x.sin h*

*h→ 0**2h*

*= lim* ***–*** *2 sin(x) sin(h)*

*h→ 0**2h*

*=* ***–*** *sin x × lim sin(h)*

*h→ 0**h*

*=* ***–*** *sin x ×* ***1*** *if x* is in radians!

This is a VERY good reason for using RADIANS when differentiating Trigonometric functions.

In fact, I think it is the only place that radians need to be used.