TRIGONOMETRIC FORMULAE.

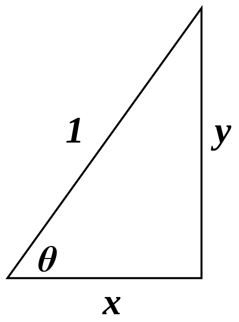
The formula sheet for New Zealand Calculus examinations contains a multitude of trigonometric formulae and despite teachers going to a lot of trouble carefully deriving these formulae, I guarantee that 99% of students do not remember where they have come from.

The most valuable and useful one concerns the trigonometric version of Pythagoras' Theorem.

$$\sin^2\theta + \cos^2\theta = 1$$

The proof is so incredibly simple yet very few students actually can recall it when asked!

The first step is to let the hypotenuse be of length 1 unit:



Using the old SOH CAH TOA idea :

$$\sin\theta = \frac{y}{1} = y$$

$$\cos\theta = \frac{x}{1} = x$$

Pythagoras' Theorem is simply $y^2 + x^2 = 1^2$

Substituting $y = \sin\theta$ *and* $x = \cos\theta$ we get $(\sin\theta)^2 + (\cos\theta)^2 = 1$

usually written as: $\sin^2\theta + \cos^2\theta = 1$

Incidentally since $\tan\theta = \frac{y}{x}$ and substituting $y = \sin\theta$ *and* $x = \cos\theta$

this is the best way to explain clearly WHY $\tan\theta = \frac{\sin\theta}{\cos\theta}$

I explained in a previous article that it is time to get rid of the reciprocal trigonometric functions (***sec x, cosec x and cot x)***.
They are a relic of past centuries and of no actual use whatsoever.
If we must actually give our calculus students the derivatives of ***sin x, cos x*** and ***tan x*** on the formula sheet then we should just give them the following version:

$y = f(x)$	$\frac{dy}{dx} = f'(x)$
$\ln(x)$	$\frac{1}{x}$
e^{ax}	ae^{ax}
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\frac{1}{\cos^2 x}$

There should be no mention of ***sec x, cosec x*** and ***cot x*** whatsoever.
I think that the derivatives of functions such as $y = (\sin x)^{-1}$ should be simply worked out and left in terms of ***sin x*** and ***cos x***.
eg $y = (\sin x)^{-1}$

$$\frac{dy}{dx} = -1\,(\sin x)^{-2}(\cos x)$$

$$= -\frac{\cos x}{(\sin x)^2}$$

A typical batch of very narrowly specialised formulae is:
Products
2sinA cosB = sin(A + B) + sin(A – B)
2cosA sinB = sin(A + B) – sin(A – B)
2cosA cosB = cos(A + B) + cos(A – B)
2sinA sinB = cos(A – B) – cos(A + B)

I suspect the **sole purpose** of these formulae is just so we can integrate things such as:

$$\int 2\cos(5x)\cos(3x)\,dx = \int \cos(8x) + \cos(2x)dx$$

$$= \frac{\sin(8x)}{8} + \frac{\sin(2x)}{2} + c$$

Frankly the time and effort needed to derive the above formulae, especially when many people do not even do a trigonometry assessment in their courses, makes it not worthwhile to bother with them at all.
These integrals are hardly crucial to a first calculus course.

I think that the **Compound Angle** formulae are probably the only other useful ones which we actually need to use.

Compound Angles

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

In order to understand how to multiply and divide **COMPLEX NUMBERS** in polar form, we need to recognise the formulae for sin(A + B) and cos(A + B)

Let $u = 1\ \mathrm{cis}(A)$ and $v = 1\ \mathrm{cis}(B)$

The PRODUCT $\boldsymbol{u \times v}$

$= (\cos A + i \sin A) \times (\cos B + i \sin B)$

$= \cos A \cos B + i(\sin A \cos B + \cos A \sin B) + \boldsymbol{i^2} (\sin A \sin B)$

$= \boldsymbol{\cos A \cos B - \sin A \sin B + i(\sin A \cos B + \cos A \sin B)}$

$= \boldsymbol{\cos(A + B) + i \sin(A + B)}$

This means that $\mathrm{cis}(30^0) \times \mathrm{cis}(45^0) = \mathrm{cis}(30 + 45) = \mathrm{cis}(75^0)$

Or $\mathrm{cis}(\pi/6) \times \mathrm{cis}(\pi/4) = \mathrm{cis}(\pi/6 + \pi/4) = \mathrm{cis}(7\pi/12)$

The QUOTIENT $\boldsymbol{\dfrac{u}{v}}$

$= \dfrac{(\cos A + i\sin A)}{(\cos B + i\sin B)} = \dfrac{(\cos A + i\sin A)}{(\cos B + i\sin B)} \times \dfrac{(\cos B - i\sin B)}{(\cos B - i\sin B)}$

$= \dfrac{\cos A \cos B + \sin A \sin B + i(\sin A \cos B - \cos A \sin B)}{\cos^2 B + \sin^2 B}$

$= \boldsymbol{\cos(A - B) + i \sin(A - B)}$

This means that $\dfrac{\mathrm{cis}(45^0)}{\mathrm{cis}(30^0)} = \mathrm{cis}(45 - 30) = \mathrm{cis}(15^0)$

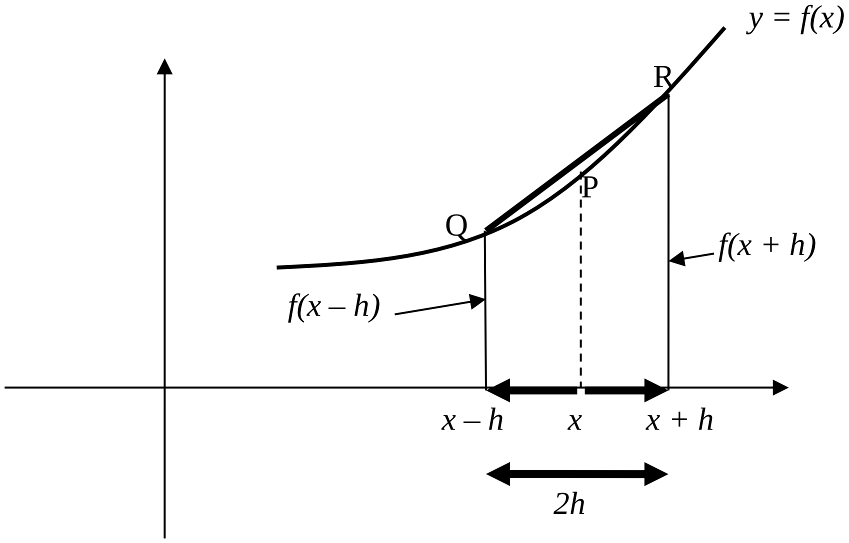
or $\dfrac{\mathrm{cis}(\pi/4)}{\mathrm{cis}(\pi/6)} = \mathrm{cis}(\pi/4 - \pi/6) = \mathrm{cis}(\pi/12)$

Imagine how elated De Moivre would have felt discovering this!

Also, in order to PROVE the derivative of ***y* = *sin x is y′* = *cos x***,
instead of using the usual "right hand" form of the derivative :

$$y' = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

I like to use the "**two sided**" form of the derivative:



The gradient of chord QR is a good approximation of the gradient of the tangent at P.

The gradient of the CHORD QR = $\dfrac{f(x+h) - f(x-h)}{2h}$

The "two sided" version of the derivative is:

Gradient at P = $\displaystyle\lim_{h\to 0} \frac{\mathbf{f(x+h) - f(x-h)}}{\mathbf{2h}}$

If ***y* = *sin x*** then $y' = \displaystyle\lim_{h\to 0} \frac{\sin(x+h) - \sin(x-h)}{2h}$

This is much easier for students to follow than the traditional method .

$$= \lim_{h\to 0} \frac{\cancel{\sin x.\cos h} + \cos x.\sin h - \cancel{\sin x.\cos h} + \cos x.\sin h}{2h}$$

$$= \lim_{h\to 0} \frac{\cancel{2}\cos(x)\ \sin(h)}{\cancel{2}\ h}$$

$$= \cos x \times \lim_{h\to 0} \frac{\sin(h)}{h}$$

$= \boldsymbol{\cos x} \times \underline{\mathbf{1}}$ ***if x* is in radians!**

(Or $= \cos x \times$ **<u>0.01745</u> if *x* is in degrees!**) see below….

Consider : $L = \lim_{h \to 0} \dfrac{\sin(h)}{h}$

USING DEGREES :

Let $h = 0.0001$ degrees

$$L \approx \frac{\sin(0.0001^0)}{0.0001}$$

$$= 0.017453292$$

So, if $y = \sin(x \text{ degrees})$

$$\frac{dy}{dx} = \cos(x) \times \underline{0.01745}$$

USING RADIANS :

Let $h = 0.0001$ rads

$$L \approx \frac{\sin(0.0001^{\text{rad}})}{0.0001}$$

$$= .999999999\ldots = \mathbf{1}$$

So, if $y = \sin(x \text{ radians})$

$$\frac{dy}{dx} = \cos(x) \times \underline{1}$$

Similarly for the derivative of $y = \cos x$

Gradient at P = $\lim_{h \to 0} \dfrac{f(x+h) - f(x-h)}{2h}$

If y = cos x then $y' = \lim_{h \to 0} \dfrac{\cos(x+h) - \cos(x-h)}{2h}$

This is much easier for students to follow than the traditional method .

$$= \lim_{h \to 0} \frac{\cancel{\cos x.\cos h} - \sin x.\sin h - \cancel{\cos x.\cos h} - \sin x.\sin h}{2h}$$

$$= \lim_{h \to 0} \frac{-\cancel{2}\sin(x)\ \sin(h)}{\cancel{2}h}$$

$$= -\sin x \times \lim_{h \to 0} \frac{\sin(h)}{h}$$

$= -\sin x \times \mathbf{1}$ *if* x is in radians!

<u>This is a VERY good reason for using RADIANS when differentiating Trigonometric functions</u>.

In fact, I think it is the only place that radians need to be used.