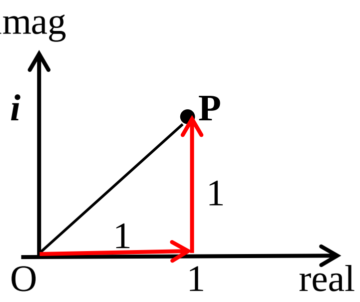
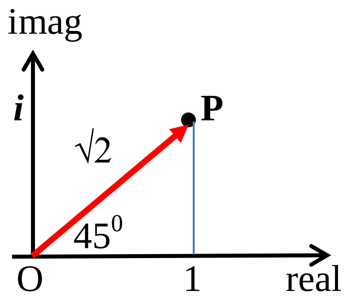
RECTANGULAR and POLAR FORM of Complex Numbers.

1.





$z = 1 + i$ ***and*** $z = \sqrt{2}\, cis(45^0)$

The above are two ways to write the same complex number.

If we examine $z = \sqrt{2}\, cis(45^0)$ we get $z = \sqrt{2}\,(\cos 45^0 + i \sin 45^0)$

$$= \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$$

$$= 1 + i$$

Clearly the two forms of ***z*** are equal.

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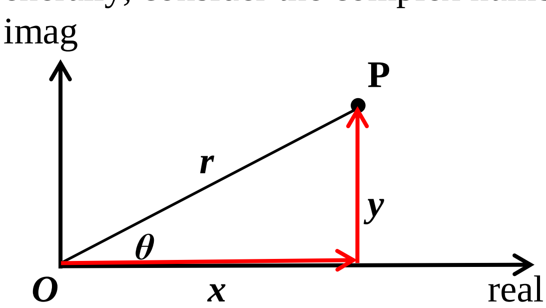
2. Generally, consider the complex number represented by **P** below:



The complex number represented by the point P on this Argand diagram can be written as:

$z = x + iy$ ***but using basic trigonometry:*** $x = r\cos\theta$ *and* $y = r\sin\theta$

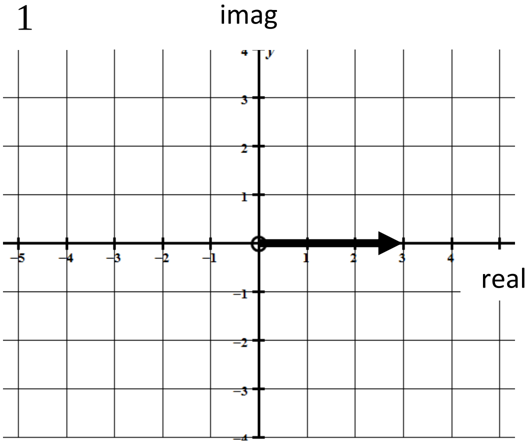
so simply substituting we get

$$z = x + iy$$
$$= r\cos\theta + i\, r\sin\theta$$
$$= r(\cos\theta + i\sin\theta)$$

Which we write in the short form $z = r\, cis\,(\theta)$

<u>IMPORTANT POINT</u>: ***r*** is the **LENGTH** of OP so it is <u>ALWAYS POSITIVE</u>.

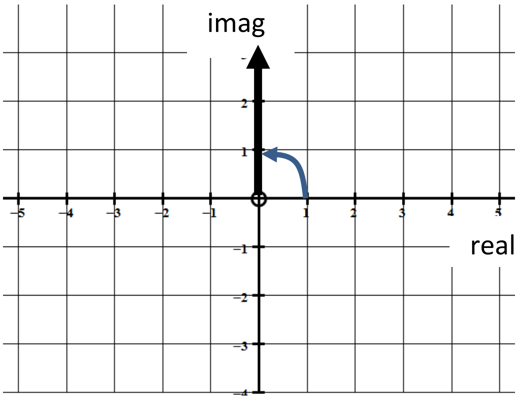
1



Here $z = 3 + 0i$ in rectangular form.
The <u>length</u> of $z = 3$
also called the <u>modulus</u> of z or $|z|$

The angle z makes with the positive real axis is 0^0
also called <u>arg(z)</u>
So in polar form $z = 3$ cis (0^0)

2

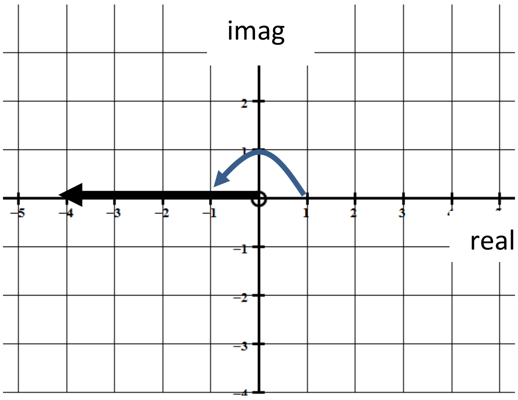


Here $z = 0 + 3i$ in rectangular form.
The <u>length</u> of z or $|z| = 3$ **(not 3i)**

The angle z makes with the positive real axis is $\arg(z) = 90^0$

So in polar form $z = 3$ cis (90^0)

3

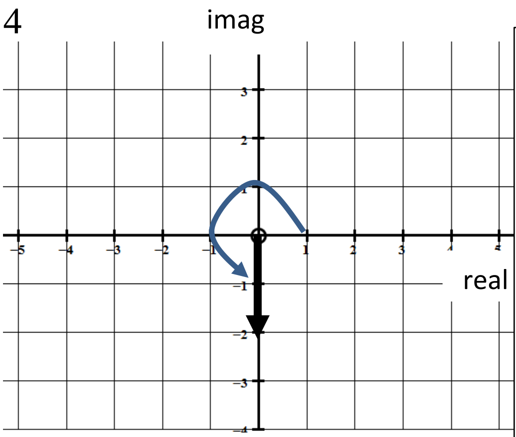


Here $z = -3 + 0i$ in rectangular form.
The <u>length</u> of z or $|z| = 4$ **(not -4)**

The angle z makes with the positive real axis is $\arg(z) = 180^0$

So in polar form $z = 4$ cis (180^0)

4

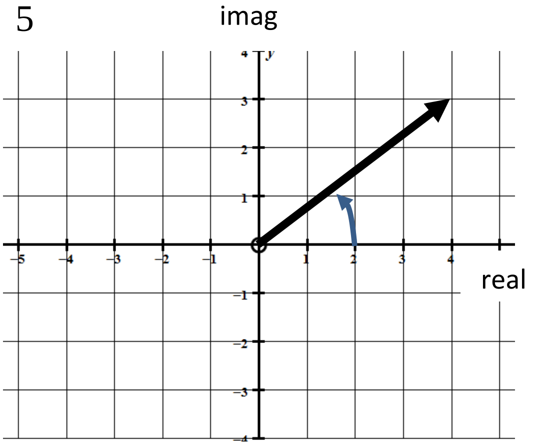


Here $z = 0 – 2i$ in rectangular form.
The <u>length</u> of z or $|z| = 2$ **(not -2i)**

The angle z makes with the positive real axis is $\arg(z) = 270^0$ or -90^0

So in polar form $z = 2$ cis (270^0)
or $z = 2$ cis(-90^0)

5

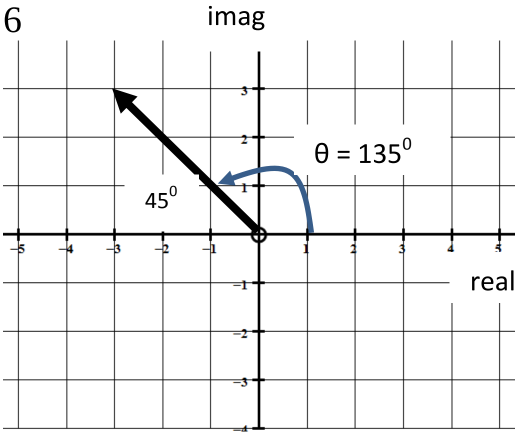


Here z = 4 + 3i in rectangular form.
The <u>length</u> of z or $|z| = \sqrt{(4^2 + 3^2)} = 5$
by Pythagoras' theorem.

The angle z makes with the positive real axis is $\tan^{-1}(3/4) \approx 36.9^0$
so $\arg(z) \approx 36.9^0$

So in polar form z = 5 cis (36.9^0)

6

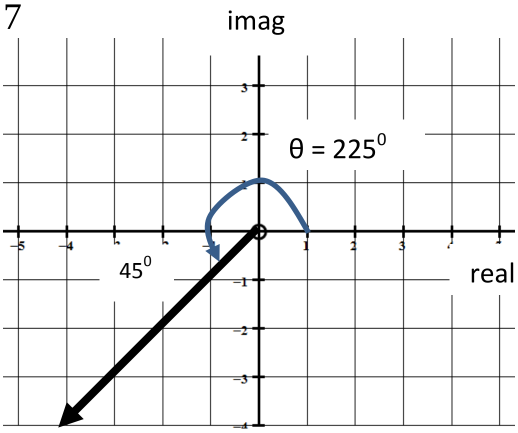


Here z = -3 + 3i in rectangular form.
The <u>length</u> of z or $|z| = \sqrt{(3^2 + 3^2)} = \sqrt{18}$
by Pythagoras' theorem.

The angle z makes with the positive real axis is $\arg(z) = 180 - 45 = 135^0$

So in polar form $z = \sqrt{18}$ cis (135^0)

7

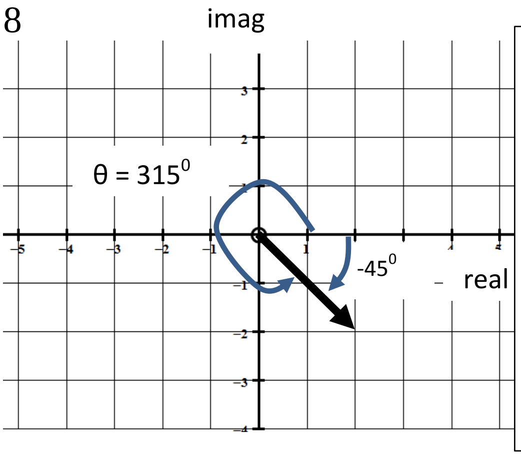


Here z = -4 – 4i in rectangular form.
The <u>length</u> of z or $|z| = \sqrt{(4^2 + 4^2)} = \sqrt{32}$

The angle z makes with the positive real axis is $\arg(z) = 225^0$ or -135^0

So in polar form $z = \sqrt{32}$ cis (225^0)
or $z = \sqrt{32}$ cis (-135^0)

8

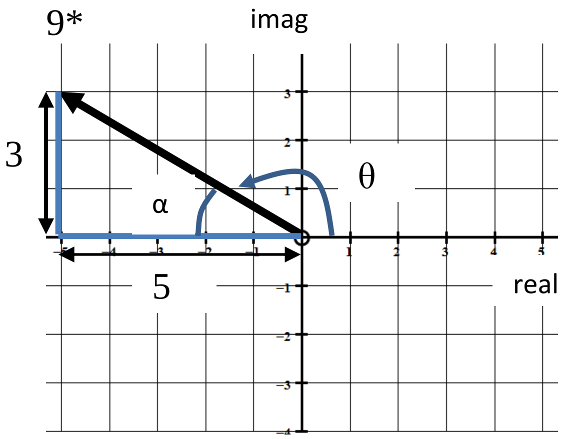


Here z = 2 – 2i in rectangular form.
The <u>length</u> of z or $|z| = \sqrt{8}$

The angle z makes with the positive real axis is $\arg(z) = 315^0$ or -45^0

So in polar form $z = \sqrt{8}$ cis (315^0)
or $z = \sqrt{8}$ cis(-45^0)

9*



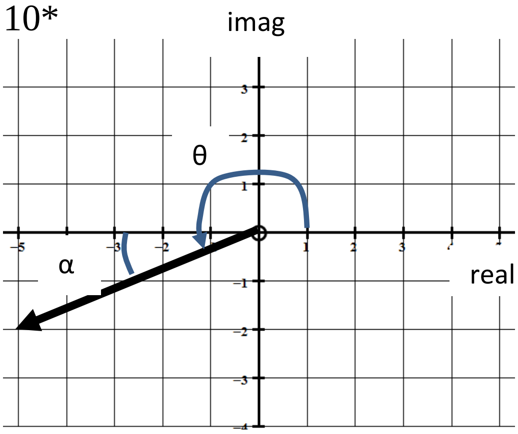
Here z = -5 + 3i in rectangular form.
The <u>length</u> of z or $|z| = \sqrt{(25+9)}$
$= \sqrt{34}$

Angle $\alpha = \tan^{-1}(3/5) \approx 31^0$

So θ, the angle z makes with the positive real axis is arg(z) = 180 – 31 = 149^0

So in polar form $z = \sqrt{34}\ \text{cis}\ (149^0)$

10*



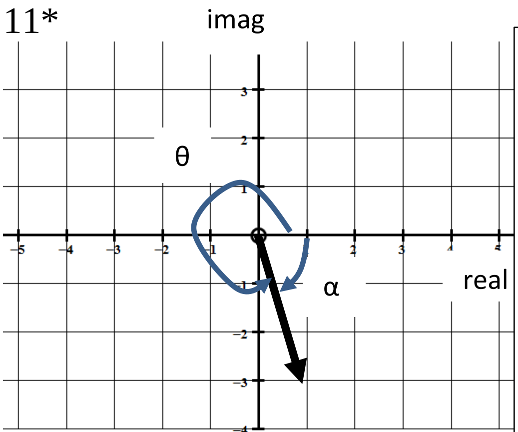
Here z = -5 – 2i in rectangular form.
The <u>length</u> of z or $|z| = \sqrt{(25+4)} = \sqrt{29}$

Angle $\alpha = \tan^{-1}(2/5) \approx 21.8^0$

So θ, the angle z makes with the positive real axis is arg(z) = 180 +201.8 = 201.8^0

So in polar form $z = \sqrt{29}\ \text{cis}\ (201.8^0)$

11*



Here z = 1 – 3i in rectangular form.
The <u>length</u> of z or $|z| = \sqrt{(9+1)} = \sqrt{10}$

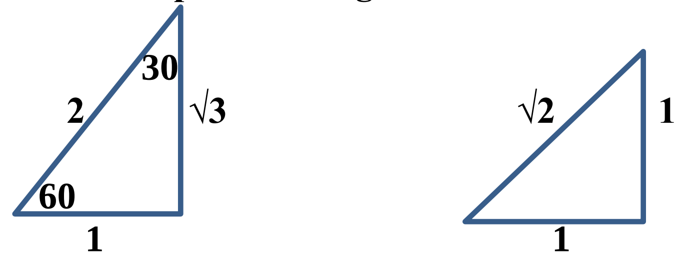
Angle $\alpha = \tan^{-1}(3/1) \approx 71.6^0$

So θ, the angle z makes with the positive real axis is arg(z) = 360 – 71.6 = 288.4^0

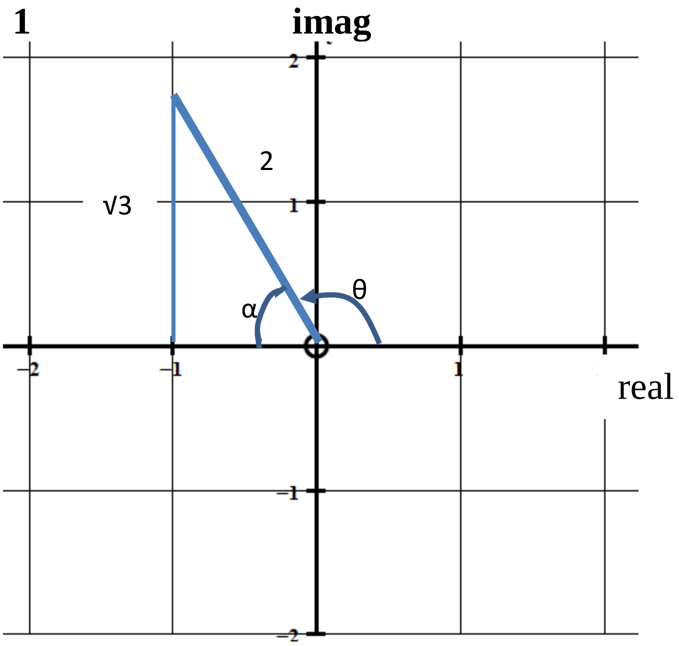
So in polar form $z = \sqrt{10}\ \text{cis}\ (288.4^0)$

SPECIAL CASES.

Look out for the "special triangles"



1



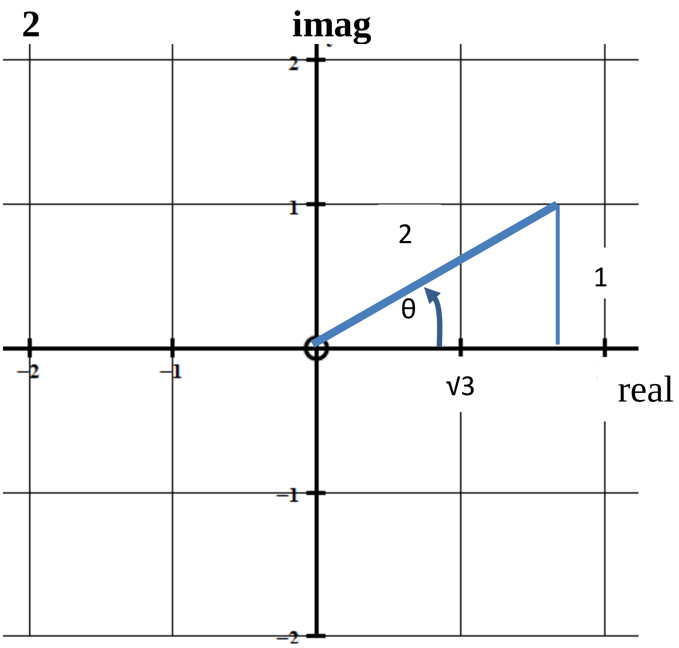
Here z = -1 + i√3
in rectangular form.
The <u>length</u> of z or |z|
= √(3 + 1) = √4 = 2

α is clearly 60⁰ (see special triangle)

or $\tan^{-1}(\sqrt{3}) = 60^0$
So $\theta = 180 - 60 = 120^0$

So in polar form $z = 2 \text{ cis } 120^0$

2



Here $z = \sqrt{3} + i$ ***in rectangular form.***
The <u>length</u> of z or |z|
= √(3 + 1) = √4 = 2

θ ***is clearly*** 30^0 (see special triangle)

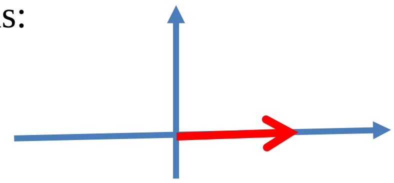
So in polar form $z = 2 \text{ cis } 30^0$

SPECIAL NOTE:

If we use a constant real number ***p***, such as ***z = p + 0i***
obviously ***p*** could be a **positive** number or a **negative** number.

This means that we cannot write ***z*** in its **polar form** unless
we know whether ***p*** is positive or negative.

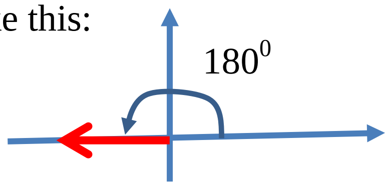
If p is a **positive** real number then ***z*** looks like this:



and ***arg(z)*** would be 0^0

so $z = p\ cis(0^0)$

But if p is a **negative** real number then ***z*** looks like this:

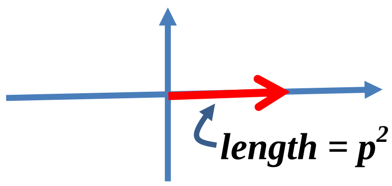


and ***arg(z)*** would be 0^0

so $z = p\ cis(180^0)$ not $-\,p\ cis(180^0)$

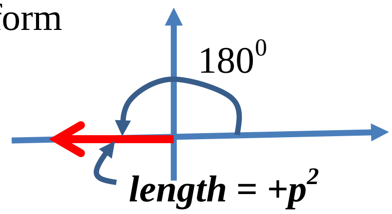
However, if we say ***p*** is any real number (positive or negative) then obviously
we could say that even powers such as p^2 *or* p^4 *or* p^6 *etc* would be POSITIVE
and any odd powers such as p^1 *or* p^3 *or* p^5 *etc* would be NEGATIVE.

Therefore if we want to express $z = p^2 + 0i$ in polar form
we can be confident that
arg(z) would be 0^0



so $z = p^2\ cis(0^0)$

And if we want to express $z = -\,p^2 + 0i$ in polar form
we can be confident that
and ***arg(z)*** would be 180^0



so $z = p^2\ cis(180^0)$ *obviously not* $-\,p^2 cis(180^0)$

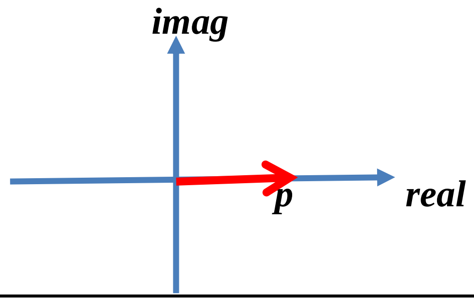
In the following examples we will assume that the variable *p* is a POSITIVE REAL NUMBER.

FURTHER SPECIAL CASES:

1. Suppose $z = p + 0i$

Clearly $|z| = p$
and $\arg(z) = 0^0$

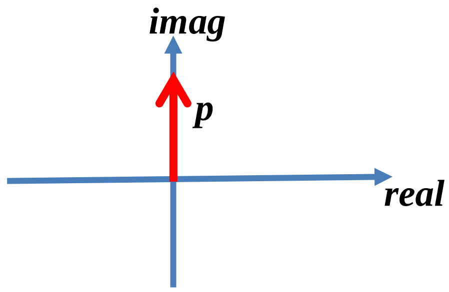
so $z = p\ cis(0^0)$



2. Suppose $z = 0 + pi$

Clearly $|z| = p$ (NOT pi)
and $\arg(z) = 90^0$

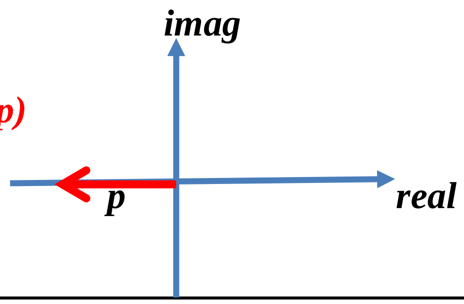
so $z = p\ cis(90^0)$



3. Suppose $z = -p + 0i$

Clearly $|z| = length = p$ (NOT –p)
and $\arg(z) = 180^0$

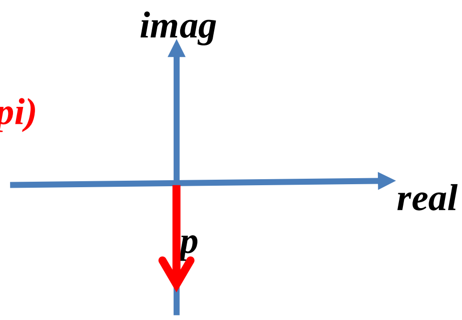
so $z = p\ cis(180^0)$



4. Suppose $z = 0 - pi$

Clearly $|z| = length = p$ (NOT –pi)
and $\arg(z) = 270^0$ or -90^0

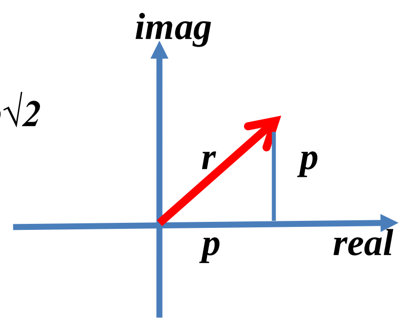
so $z = p\ cis(270^0)$



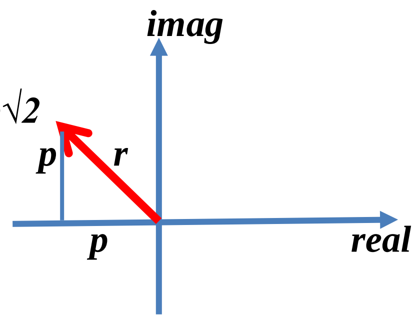
5. Suppose $z = p + pi$

Clearly $|z| = \sqrt{(p^2 + p^2)} = \sqrt{2p^2} = p\sqrt{2}$
and $\arg(z) = 45$

so $z = p\sqrt{2}\ cis(45^0)$



6. Suppose $\boldsymbol{z = -p + pi}$



Clearly $|z| = \sqrt{(p^2 + p^2)} = \sqrt{2p^2} = p\sqrt{2}$
and $\arg(z) = 135^0$

so $z = p\sqrt{2}\ \mathrm{cis}(135^0)$

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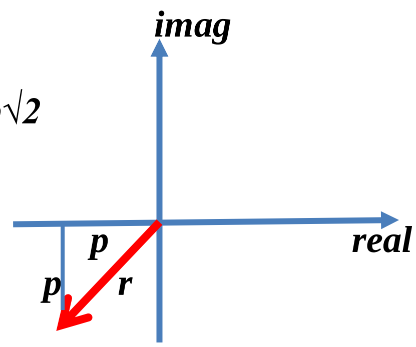
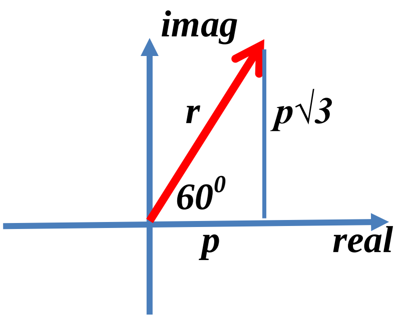
7. Suppose $\boldsymbol{z = -p - pi}$



Clearly $|z| = \sqrt{(p^2 + p^2)} = \sqrt{2p^2} = p\sqrt{2}$
and $\arg(z) = 225^0$

so $z = p\sqrt{2}\ \mathrm{cis}(225^0)$

8*. Suppose $\boldsymbol{z = p + (p\sqrt{3})i}$



Clearly $|z| = r = \sqrt{(p^2 + 3p^2)}$
$= \sqrt{4p^2}$
$= 2p$

and $\arg(z) = \tan^{-1}\left(\dfrac{p\sqrt{3}}{p}\right) = 60^0$

so $z = 2p\ \mathrm{cis}(60^0)$

This triangle should be recognised as one of our special triangles.

Always look out for the "special triangles"