DE MOIVRE'S DISCOVERY.

Considering operations with complex numbers in the Cartesian (Rectangular) form : eg let $u = 2 + 3i$ and $v = 1 + i$

then the operations $u + v$ and $u - v$ are very straightforward.

ie $u + v = 3 + 4i$ and $u - v = 1 + 2i$

but the operations $u \times v$ and $u \div v$ are disappointingly quite awkward.

ie $uv = (2 + 3i)(1 + i) = 2 + 2i + 3i + 3i^2 = -1 + 5i$

and $\dfrac{u}{v} = \dfrac{(2 + 3i)}{(1 + i)} = \dfrac{(2 + 3i)}{(1 + i)} \times \dfrac{(1 - i)}{(1 - i)} = \dfrac{5}{2} + \dfrac{i}{2}$

De Moivre decided to try multiplying and dividing in the POLAR FORM and discovered something amazingly neat!

Let $u = 1 \text{ cis}(A)$ and $v = 1 \text{ cis}(B)$

The PRODUCT $u \times v$

$= (\cos A + i \sin A) \times (\cos B + i \sin B)$

$= \cos A \cos B + i(\sin A \cos B + \cos A \sin B) + i^2 (\sin A \sin B)$

$= \mathbf{\cos A \cos B - \sin A \sin B + i(\sin A \cos B + \cos A \sin B)}$

$= \mathbf{\cos(A + B) + i \sin(A + B)}$

This means that $\text{cis}(30^0) \times \text{cis}(45^0) = \text{cis}(30 + 45) = \text{cis}(75^0)$

Or $\text{cis}(\pi/6) \times \text{cis}(\pi/4) = \text{cis}(\pi/6 + \pi/4) = \text{cis}(7\pi/12)$

The QUOTIENT $\dfrac{u}{v}$

$= \dfrac{(\cos A + i\sin A)}{(\cos B + i\sin B)} = \dfrac{(\cos A + i\sin A)}{(\cos B + i\sin B)} \times \dfrac{(\cos B - i\sin B)}{(\cos B - i\sin B)}$

$= \dfrac{\cos A \cos B + \sin A \sin B + i(\sin A \cos B - \cos A \sin B)}{\cos^2 B + \sin^2 B}$

$= \mathbf{\cos(A - B) + i \sin(A - B)}$

This means that $\dfrac{\text{cis}(45^0)}{\text{cis}(30^0)} = \text{cis}(45 - 30) = \text{cis}(15^0)$

or $\dfrac{\text{cis}(\pi/4)}{\text{cis}(\pi/6)} = \text{cis}(\pi/4 - \pi/6) = \text{cis}(\pi/12)$

Imagine how elated De Moivre would have felt discovering this!!!

(However, addition and subtraction are much easier in rectangular form!)

Obviously if $u = 8cis(70^0)$ and $v = 2cis(30^0)$
then $uv = 8\times2\times cis(70)\times cis(30)$
$= 16\ cis(100^0)$

and $\dfrac{u}{v} = \dfrac{8\ cis(70)}{2\ cis(30)} = 4\ cis(40^0)$

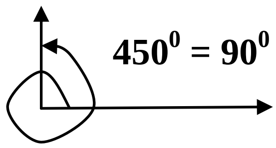
<u>De Moivre went on to consider POWERS of complex numbers.</u>
If $u = 3cis(20^0)$
then $u^2 = 3^2\ cis(20)\times cis(20)$
$= 9\ cis\ (2 \times 20)$
$= 9\ cis\ (40^0)$

So obviously $u^3 = 3^3\ cis(20)\times cis(20)\ \times cis(20)$
$= 27\ cis\ (3 \times 20)$
$= 27\ cis\ (60^0)$

Similarly $u^4 = 3^4 cis(4\times20)$
$= 81\ cis\ (80^0)$

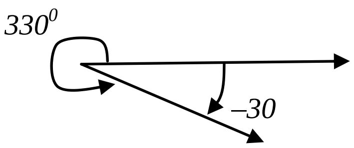
De Moivre generalised this and called it his own THEOREM.
De Moirvre's Theorem
If $z = r\ cis(\theta)$ then $z^n = r^n\ cis(n\theta)$

<u>Special Application</u>. Find $(1 + i)^{10}$
NOTE :$(1 + i) = \sqrt{2}\ cis(45^0) = 2^{½}\ cis\ 45$



So $(1 + i)^{10} = (2^{½})^{10}\ cis\ (10\times45)$
$= 2^5\ cis\ (450^0)$
$= 32\ cis(90^0)$
$= 32(cos90 + isin90)$
$= 32(0 + 1i)$
$= 32i$

It is MOST useful to be familiar with the SPECIAL TRIANGLES!
eg Find $(\sqrt{3} - i)^6$



$=2^6 cis(6\times330) or\ 2^2 cis(6\times-30)$
$=64\ cis(1980)\ or\ 64cis(-180)$
$=64(cos-180 + isin-180)$
$=64(-1 + 0i)$
$= -64$