## SOLVING COMPLEX NUMBER EQUATIONS OF THE FORM $z^n = a + ib$

Students need to realise that the ***equation*** $ax + b = 4x + 6$ can be "solved" in the usual way and get $x = \dfrac{6 - b}{a - 4}$

but the ***identity*** $ax + b = 4x + 6$ just implies $a = 4$ and $b = 6$.

Similarly if $a + bi = 5 + 2i$ then $a = 5$ and $b = 2$.
In particular, if $r(\cos\theta + i\sin\theta) = 4(\cos 30^0 + i\sin 30^0)$ then $r = 4, \theta = 30^0$.

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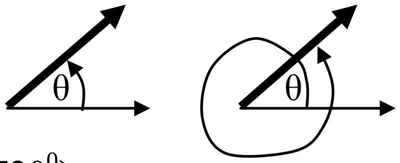
A basic trigonometric idea is that the directions θ, θ + $360^0$, θ + $720^0$ , …….
are all the same.




In particular, $\sin(\theta) = \sin(\theta + 360^0) = \sin(\theta + 720^0)$ etc
or in general $\underline{\sin(\theta) = \sin(\theta + 360n)}$ ie any number of 360's can be added.

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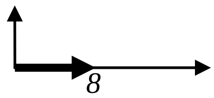
**In all cases we put *z = r cis(θ)* and use De Moivre's Theorem.**

1. **Solve** $z^3 = 8$

*NOTE: 8 = 8 + 0i*




*So in polar form r = 8 and θ = 0*
*8 + 0i = 8 cis(0) but in this case we need*
*8 + 0i = 8 cis(0 + 360n)*

$$\text{So} \quad [r\,cis(\theta)]^3 = 8\,cis(0 + 360n)$$

$$r^3\,cis(3\theta) = 8\,cis(360n)$$

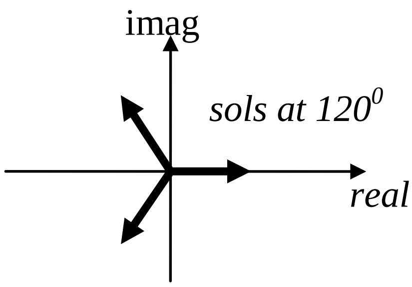
$$\text{so } r^3 = 8 \quad \text{and} \quad 3\theta = 360n$$

producing $r = 2$ and $\theta = 120n = 0^0, 120^0, 240^0$
(we only need 3 angles because $z^3 = 8$ only has 3 solutions)

$z_1 = 2\,cis(0)$

$z_2 = 2\,cis(120^0)$

$z_3 = 2\,cis(240^0)$




Sometimes we may be required to express answers in a + bi form.

$z_1 = 2\,cis(0) = 2(\cos 0 + i\sin 0) = 2 + 0i$

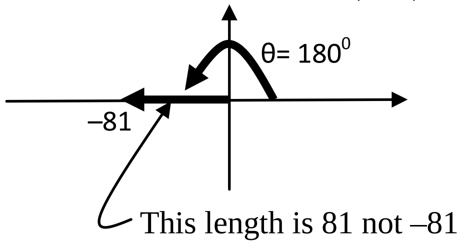
$z_2 = 2\,cis(120^0) = 2\cos(120) + i2\sin(120) = -1 + i\sqrt{3}$

$z_3 = 2\,cis(240^0) = 2\cos(240) + i2\sin(240) = -1 - i\sqrt{3}$

2. **Solve** $\boldsymbol{z^4 = -81}$

Students MUST grasp the idea that the **modulus** is always **POSITIVE**.
It is just a LENGTH.

NOTE : $-81 + 0i = +81 \text{ cis } (180^0)$




$z^4 = +\mathbf{81} \text{ cis}(180 + 360n)$

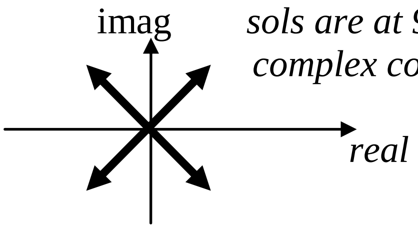
$r^4 \text{cis}(4\theta) = 81 \text{ cis}(180 + 360n)$

$r^4 = 81$ and $4\theta = 180 + 360n$

*so* $r = 3$ (NEVER ±3) and $\theta = 45 + 90n = 45, 135, 225, 315$
(we now need 4 angles because $z^4 = -81$ has 4 solutions)

*sols are at* $90^0$ *and complex conjugates*

$z_1 = 3 \text{ cis}(45)$
$z_2 = 3 \text{ cis}(135^0)$
$z_3 = 3 \text{ cis}(225^0)$
$z_4 = 3 \text{ cis}(315^0)$

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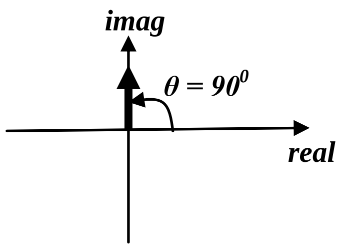
3. Solve $\boldsymbol{z^4 = i}$

**NOTE:** $0 + i = 1 \text{ cis}(90^0)$




$r^4 \text{ cis } 4\theta = 1\text{cis } (90 + 360n)$

$r = 1$ $4\theta = 90 + 360n$

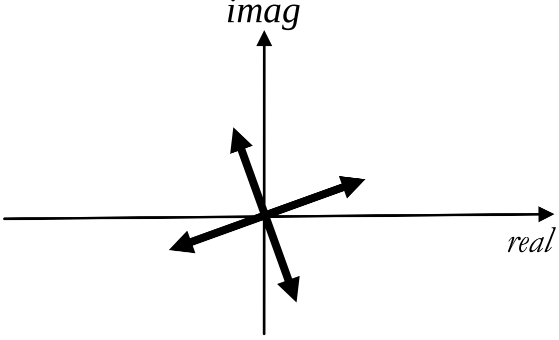
$\theta = 22.5 + 90n$

$z_1 = \text{cis } 22.5$

$z_2 = \text{cis } 112.5$

$z_3 = \text{cis } 202.5$

$z_4 = \text{cis } 292.5$




Notice that solutions are at <u>$90^0$ *to each other*</u> but are <u>*not complex conjugates*</u>.
**Solutions are only complex conjugates when all numbers in the equation are real numbers**. In this case $\boldsymbol{z^4 = i}$ which contains a non real number.

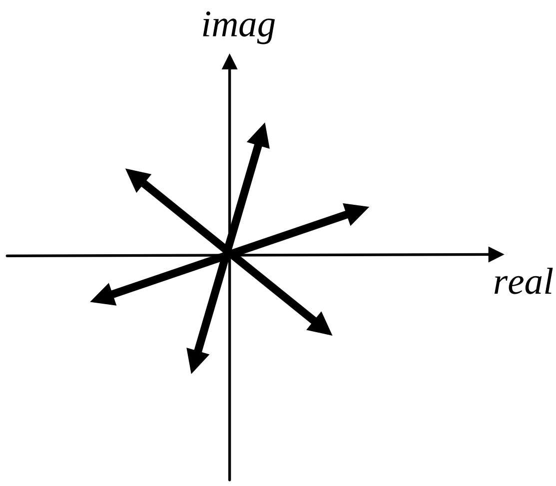
4. Solve $\boldsymbol{z^6 = 64 \text{ cis} \left(\frac{2\pi}{3}\right)}$ **or** $\boldsymbol{z = 64 \text{ cis } (120^0)}$

This equation has 6 solutions so we need 6 angles.

$r^6 \text{cis } 6\theta = 64 \text{ cis } (120 + 360n)$

$r = 2 \quad 6\theta = 120 + 360n$

$\theta = 20 + 60n$




$z_1 = 2\text{cis } 20^0$
$z_2 = 2\text{cis } 80^0$
$z_3 = 2\text{cis } 140^0$
$z_4 = 2\text{cis } 200^0$
$z_5 = 2\text{cis } 260^0$
$z_1 = 2\text{cis } 320^0$

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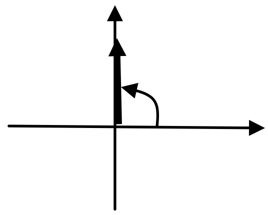
5. **Solve the equation** $\boldsymbol{z^3 - 64i = 0}$ **where** $\boldsymbol{z}$ **is a complex number** $\boldsymbol{r \text{ cis}(\theta)}$

Leave your solutions in the polar form.

(*We change the equation to the form*: $\boldsymbol{z^3 = 64i}$ )



**Let** $\boldsymbol{z = r \text{ cis } \theta}$

$\boldsymbol{z^3 = r^3 \text{ cis } 3\theta = 64 \text{ cis } (90 + 360n)}$

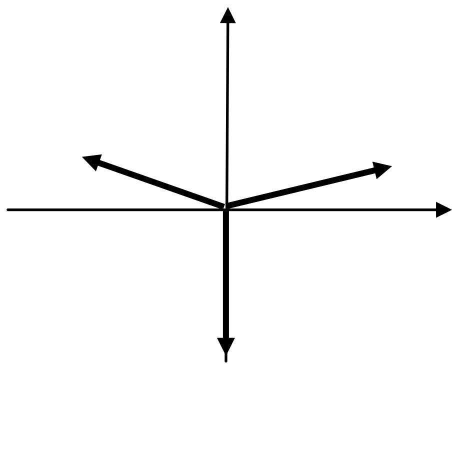
$\boldsymbol{r = 4 \quad 3\theta = 90 + 360n}$

$\boldsymbol{\theta = 30 + 120n}$

$\boldsymbol{z_1 = 4 \text{ cis } 30^0}$

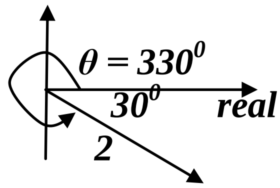
$\boldsymbol{z_2 = 4 \text{ cis } 150^0}$

$\boldsymbol{z_3 = 4 \text{ cis } 270^0 \text{ or } 4 \text{ cis } (-90^0)}$



6. Solve $z^3 = \sqrt{3} - i$ and write your answers in the form $r \text{ cis } \theta$

***A good understanding of the "special triangles" is important in cases like this!***

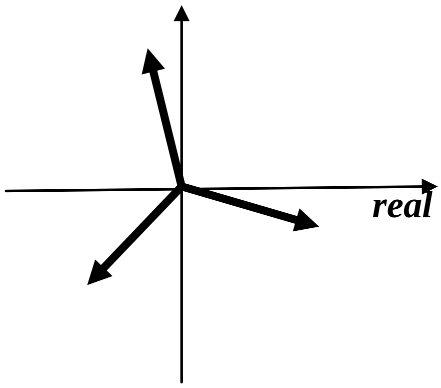



*Let* $z = r \text{ cis } \theta$

$$z^3 = r^3 \text{ cis } 3\theta = 2 \text{ cis } (330^0 + 360n)$$

$$r^3 = 3 \qquad 3\theta = 330^0 + 360n$$

$$r = 3^{1/3} \qquad \theta = 110^0 + 120n = 110^0,\ 230^0,\ 350^0$$




$z_1 = 2^{1/3} \text{ cis } 110^0$

$z_2 = 2^{1/3} \text{ cis } 230^0$ or $2^{1/3} \text{ cis}(-130^0)$

$z_3 = 2^{1/3} \text{ cis } 350^0$ or $2^{1/3} \text{ cis } (-10^0)$

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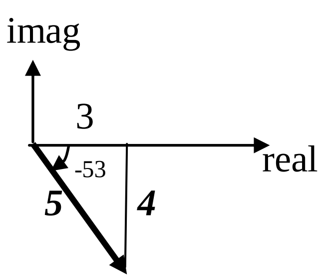
7. **Solve the equation :** $z^2 = 3 - 4i$

Note: $3 - 4i = 5\text{cis}(-53.1)$




*Let* $z = r\text{cis}\theta$

$$r^2 \text{cis } 2\theta = 5 \text{ cis}(-53.1 + 360n)$$

$$r = \sqrt{5} \qquad 2\theta = -53.1 + 360n \qquad \theta = -26.6 \text{ or } -26.6 + 180$$

$$\theta = -26.6 \text{ or } 153.4$$

***Solutions are :***

$z_1 = \sqrt{5}\text{cis}(-26.6^0)$

$z_2 = \sqrt{5}\text{cis}(153.4^0)$