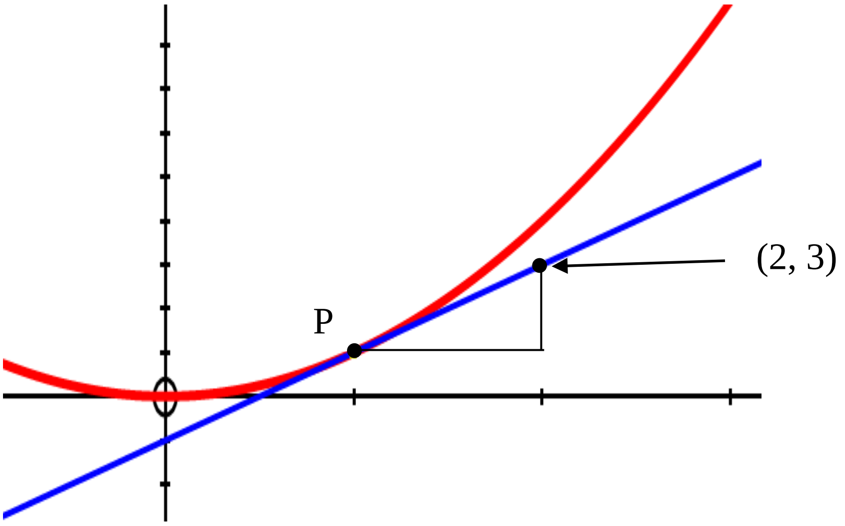
FINDING THE EQUATION OF A TANGENT TO A CURVE FROM A POINT NOT ON THE CURVE.

1. Find the equation of the tangent to $y = x^2$ from the point (2, 3)



In questions like this we do not know the coordinates of the point P where the tangent touches the curve.
If we let the x coordinate of P be a then the y coordinate will be a^2 *(ie $y = x^2$)*
Using the gradient triangle drawn above we can say that the gradient of the tangent using points (2, 3) and (a, a^2) will be $\dfrac{3 - a^2}{2 - a}$

We can find another value for the gradient at P by differentiating $y = x^2$
Gradient $y' = 2x$ and if $x = a$ at P then the gradient of the tangent is $2a$
Equating these two values we get : $\quad 2a = \dfrac{3 - a^2}{2 - a}$

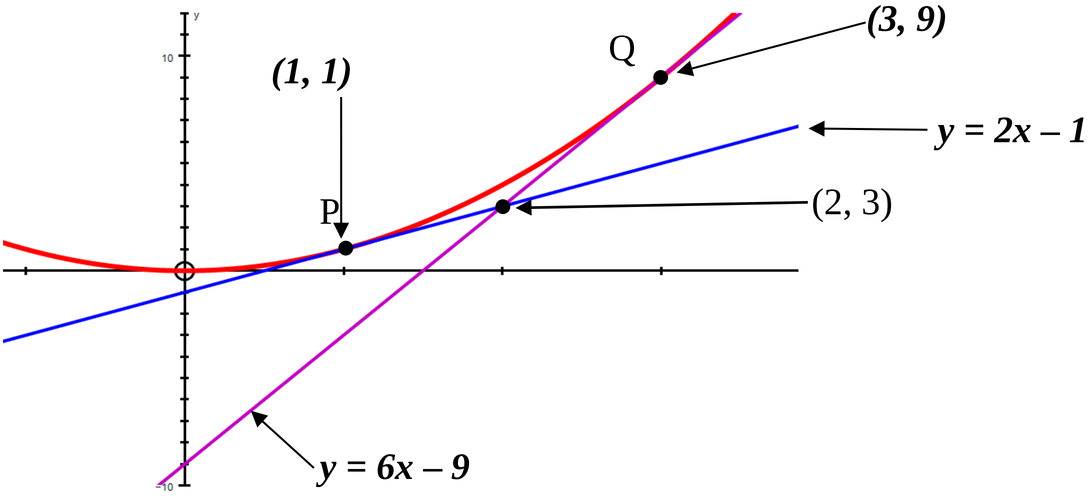
So that

$$2a(2 - a) = 3 - a^2$$
$$4a - 2a^2 = 3 - a^2$$
$$0 = a^2 - 4a + 3$$
$$0 = (a - 1)(a - 3)$$
$$a = 1 \text{ OR } a = 3$$

This means there are TWO possible tangents from (2, 3) to the curve.

If $a = 1$ then P would be (1, 1), the gradient would be $m = 2$ and using $y = mx + c$ substituting $m = 2$, $x = 1$, $y = 1$ we get $c = -1$
The tangent drawn in the above diagram is $y = 2x - 1$

If $a = 3$ then P would be (3, 9), the gradient would be $m = 6$ and using $y = mx + c$ substituting $m = 6$, $x = 3$, $y = 9$ we get $c = -9$
The other tangent drawn in the diagram below is $y = 6x - 9$



This shows the TWO tangents from (2, 3) to the parabola $y = x^2$

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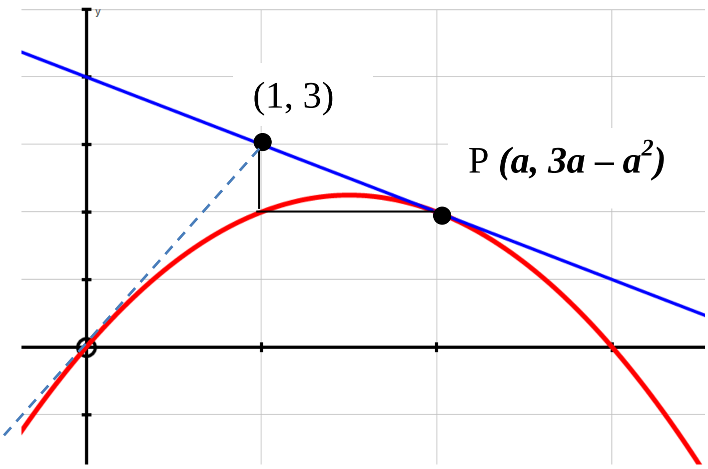
2. Find the equation of the tangents to $y = 3x - x^2$ from the point (1, 3)



Let P be the point where the tangent from (1, 3) touches the curve so the coordinates of P, are ***(a, 3a – a²).***

Using the gradient triangle shown, the gradient of the tangent in terms of ***a*** is:

$$-\frac{(3-3a+a^2)}{(a-1)}$$

Differentiating y' = 3 – 2x = 3 – 2a so equating these forms:

$$3 - 2a = -\frac{(3-3a+a^2)}{(a-1)}$$

$$3a - 3 - 2a^2 + 2a = 3a - 3 - a^2$$

$$0 = a^2 - 2a$$

So a = 0 or 2

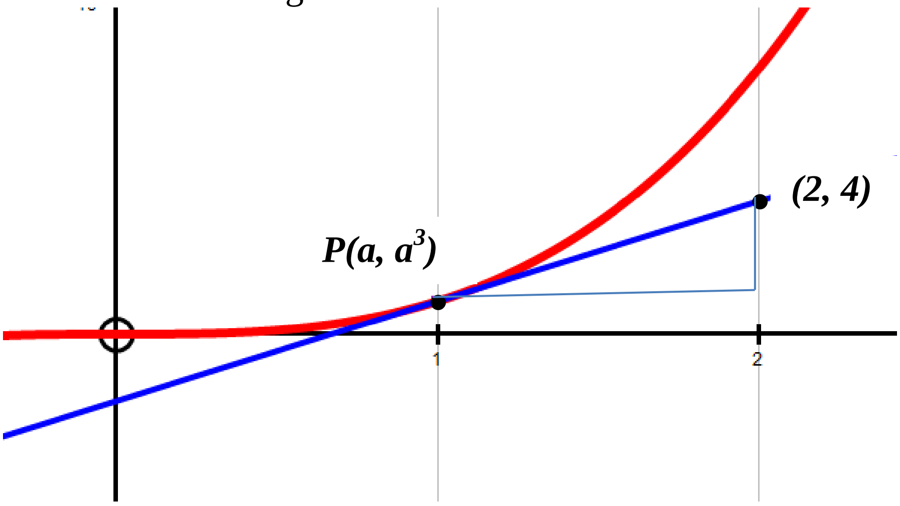
If a = 2 then P is (2, 2) and the tangent equation is y = – x + 4

If a = 0 then P is (0, 0) and the other tangent equation is y = 3x **(*see dotted line*)**

3. Find the equation of the tangents to $y = x^3$ from the point (2, 4)

It will become apparent that there are actually THREE tangents!

Consider the tangent drawn below:



Let the coordinates of the point where the tangent meets the curve be (a, a^3)

Using the gradient triangle, the gradient of the tangent is $\dfrac{4 - a^3}{2 - a}$

Differentiating, we get $y' = 3x^2 = 3a^2$

Equating these two expressions:

$$3a^2 = \frac{4 - a^3}{2 - a}$$

$$6a^2 - 3a^3 = 4 - a^3$$

$$0 = 2a^3 - 6a^2 + 4$$

$$0 = a^3 - 3a^2 + 2$$

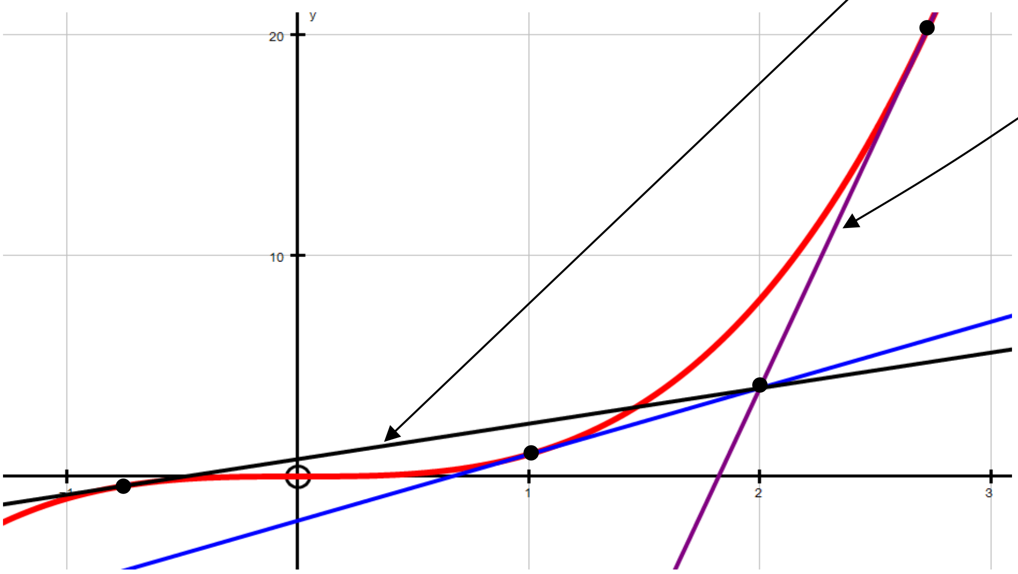
$$0 = (a - 1)(a^2 - 2a - 2)$$

So that a = 1 or 1±√3

If a = 1, P is (1, 1) the gradient is 3 and the tangent is y = 3x – 2

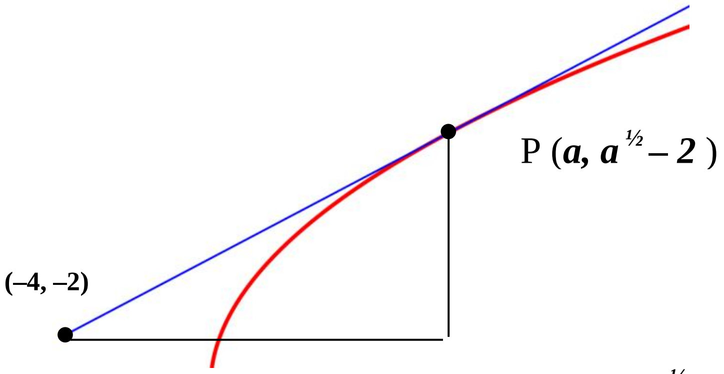
If a = 1+√3, P is (1+√3, 20.39) the grad is 22.4 and the tan is y ≈ 22.4x – 40.8

If a = 1–√3, P is (1–√3, –0.39) the grad is 1.6 and the tan is y ≈ 1.6x –0.78



4. Find the equation of the tangent to $y = x^{½} – 2$ from the point (–4, –2)

The graph is like this diagram…



Let the tangent meet the curve at $x = a$ *so* $y = a^{½} – 2$

From the gradient triangle, the gradient of the tangent = $\dfrac{a^{½} – 2 + 2}{a + 4} = \dfrac{a^{½}}{a + 4}$

If $y = x^{½} – 2$ then $y' = \dfrac{1}{2x^{½}}$ *so at* $x = a$*, the gradient of the tan is* = $\dfrac{1}{2a^{½}}$

Equating these two expressions we get: $\dfrac{a^{½}}{a + 4} = \dfrac{1}{2a^{½}}$

hence $2a = a + 4$

so that $a = 4$

The coordinates of P are $(4, 0)$

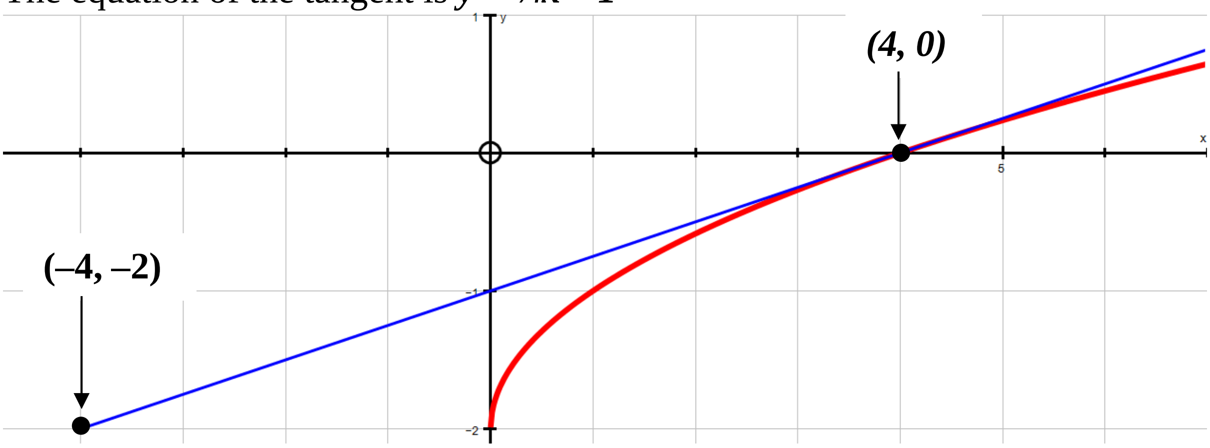
The gradient of the tangent = $\dfrac{1}{4}$

The equation of the tangent is of the form $y = mx + c$

So substituting $x = 4, y = 0, m = ¼$

we get $0 = 1 + c$ *so* $c = –1$

The equation of the tangent is $y = ¼x – 1$



4. Find the equation of the tangents to $y = x^4 - 6x^2$ from the point (0, 3)

Let the tangent meet the curve at $\boldsymbol{x = a}$ **so** $\boldsymbol{y = a^4 - 6a^2}$

From the gradient triangle, the gradient of the tangent = $\dfrac{\boldsymbol{a^4 - 6a^2 - 3}}{\boldsymbol{a}}$

If $\boldsymbol{y = y = x^4 - 6x^2}$ then $\boldsymbol{y' = 4x^3 - 12x}$
so at $\boldsymbol{x = a}$, the gradient of the tangent is = $\boldsymbol{4a^3 - 12a}$

Equating these two expressions we get: $\dfrac{\boldsymbol{a^4 - 6a^2 - 3}}{\boldsymbol{a}} = \boldsymbol{4a^3 - 12a}$

so $\boldsymbol{a^4 - 6a^2 - 3 = 4a^4 - 12a^2}$

hence $\boldsymbol{0 = 3a^4 - 6a^2 + 3}$
$\boldsymbol{0 = 3(a^2 - 1)(a^2 - 1)}$
So $\boldsymbol{a = 1}$ **or** $\boldsymbol{-1}$

<u>If $\boldsymbol{a = 1}$:</u>
The coordinates of P are $\boldsymbol{(1, -5)}$
The gradient of the tangent $\boldsymbol{= -8}$
The equation of the tangent is of the form $\boldsymbol{y = mx + c}$
So substituting $\boldsymbol{x = 1, y = -5, m = -8}$ we get $\boldsymbol{-5 = -8 + c}$ **so** $\boldsymbol{c = 3}$
The equation of the tangent is $\boldsymbol{y = -8x + 3}$
<u>If $\boldsymbol{a = -1}$:</u>
The coordinates of P are $\boldsymbol{(-1, -5)}$
The gradient of the tangent $\boldsymbol{= +8}$
The equation of the tangent is of the form $\boldsymbol{y = mx + c}$
So substituting $\boldsymbol{x = -1, y = -5, m = +8}$ we get $\boldsymbol{-5 = -8 + c}$ **so** $\boldsymbol{c = 3}$
The equation of the tangent is $\boldsymbol{y = +8x + 3}$